Testing for Multifractality and Multiplicativity using Surrogates

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Motivating Questions

- Multifractality has been reported in several hydrologic variables (rainfall, streamflow, soil moisture etc.)
- Questions of interest:
    - ✓ What is the nature of the underlying dynamics?
    - ✓ What is the simplest model consistent with the observed data?
    - ✓ What can be inferred about the underlying mechanism giving rise to the observed series?

Precipitation: Linear or nonlinear dynamics?

- Multiplicative cascades (MCs) have been assumed for rainfall motivated by a turbulence analogy (e.g., Lovejoy and Schertzer, 1991 and others)
- Recently, Ferraris et al. (2003) have attempted a rigorous hypothesis testing. They concluded that:
  - ✓ MCs are not necessary to generate the scaling behavior found in rain
  - ✓ The multifractal behavior of rain can be originated by a nonlinear transformation of a linearly correlated stochastic process.

Methodology

- Test null hypothesis:
  - ✓ H_0: Observed multifractality is generated by a linear process
  - ✓ H_1: Observed multifractality is rooted in nonlinear dynamics
- Compare observed rainfall series to “surrogates”
- Surrogates destroy the nonlinear dynamical correlations by phase randomization, but preserve all other properties (Thieler et al., 1992)

Purpose of this work

- Introduce more discriminatory metrics which can depict the difference between processes with non-linear versus linear dynamics

- Illustrate methodology on generated sequences (FIC and RWC) and establish that "surrogates" of a pure multiplicative cascade lack long-range dependence and are monofractals

- Test high-resolution temporal rainfall and make inferences about possible underlying mechanism

Metrics

1. WTMM Partition function: q = 1, 2, 3 …

$$Z(q,a) = \sum_{\mathbb{L}(a)} |T_a(x)|^q \qquad \mathbb{L}(a) - \text{set of maxima lines at scale } a$$

2. Cumulants $C_n(a)$ vs. a

$$C_1(a) \equiv \langle \ln|T_a|\rangle \sim c_1 \ln(a)$$

$$C_2(a) \equiv \langle \ln^2|T_a|\rangle - \langle \ln|T_a|\rangle^2 \sim c_2 \ln(a)$$

$$C_3(a) \equiv \langle \ln^3|T_a|\rangle - 3\langle \ln^2|T_a|\rangle\langle \ln|T_a|\rangle + \langle \ln|T_a|\rangle^3 \sim c_3 \ln(a)$$

etc.

Recall
$$\tau(q) = -c_0 + c_1 q - c_2 \frac{q^2}{2} + \cdots$$

$$D(h) = \min_q \left(qh - \tau(q) \right)$$

3. Two-point magnitude correlation analysis

$$C(a,\Delta x) = \left\langle \left(\ln|(T_a(x)| - \overline{\ln|(T_a(x)|} \right) \left(\ln|(T_a(x+\Delta x)| - \overline{\ln|(T_a(x+\Delta x)|} \right) \right\rangle$$

$$C(a,\Delta x) \sim \ln \Delta x, \qquad \Delta x > a \Rightarrow \text{long-range dependence}$$

$$C(a,\Delta x) \sim -c_2 \ln \Delta x \qquad \Rightarrow \text{multiplicative cascade}$$

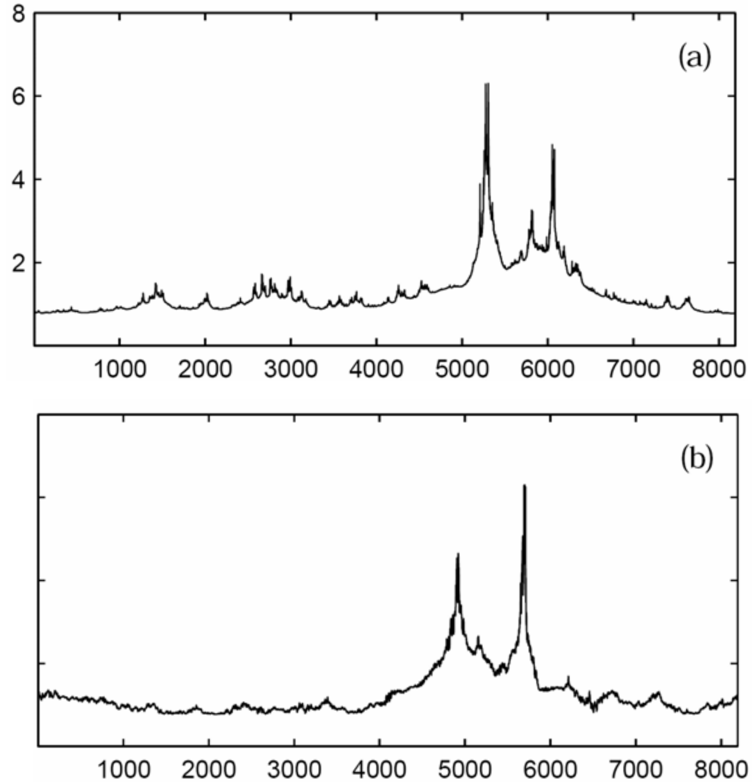
$$\left(C_2(a) \sim -c_2 \ln a \right)$$

Surrogate of an FIC

a) FIC: $c_1 = 0.13$; $c_2 = 0.26$; $H^* = 0.51$

(To imitate rain: $c_1 = 0.64$; $c_2 = 0.26$)

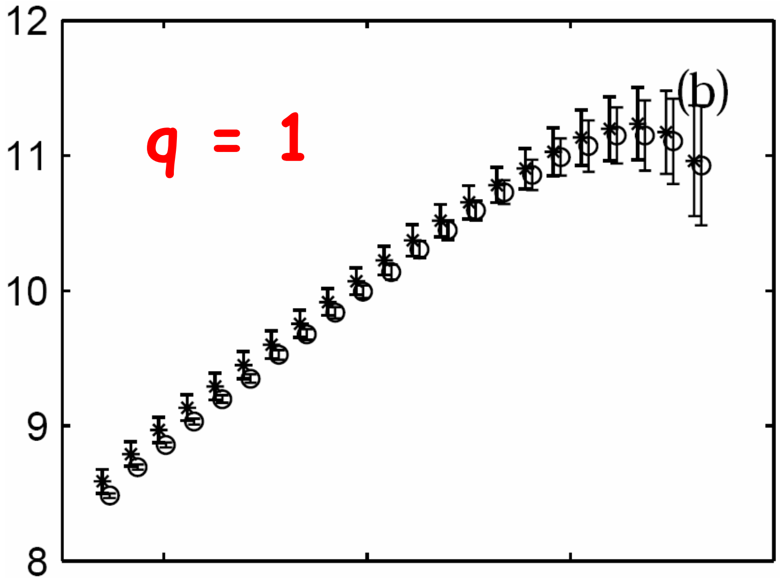
b) Surrogates

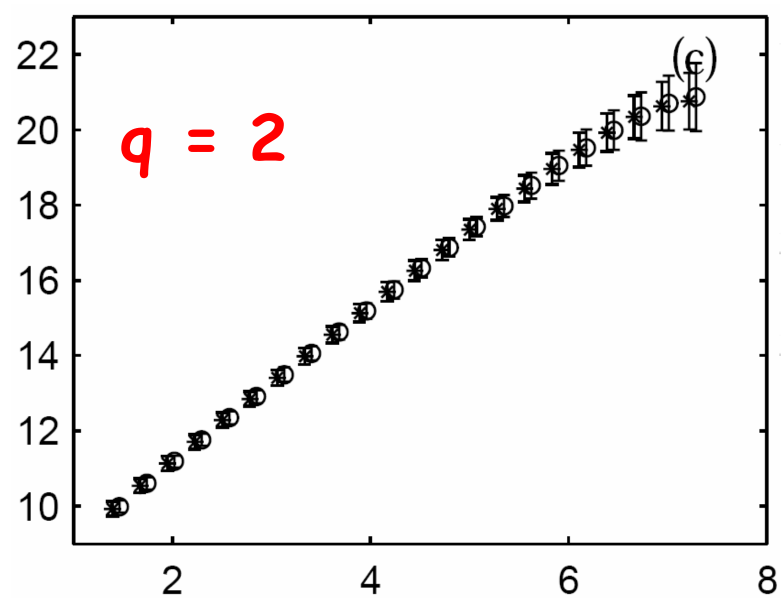


FIC

Surrogate

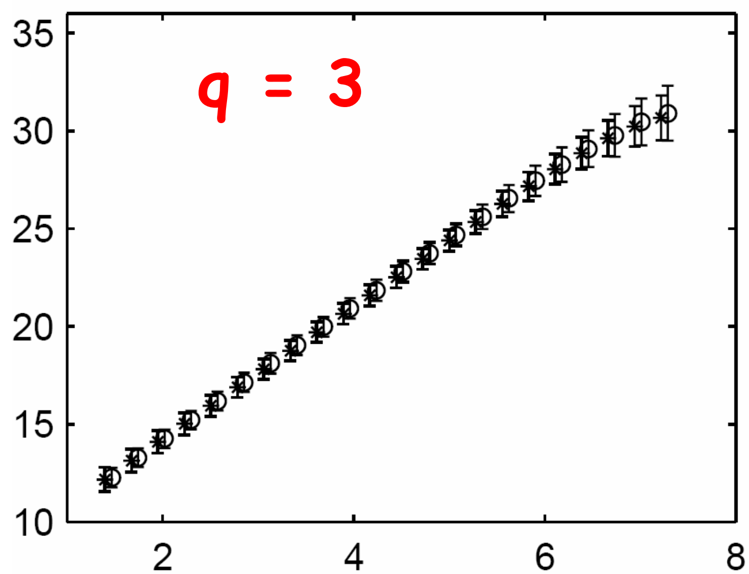
Multifractal analysis of FIC and surrogates
(Ensemble results)





ln [Z(q,a)]

ln (a)

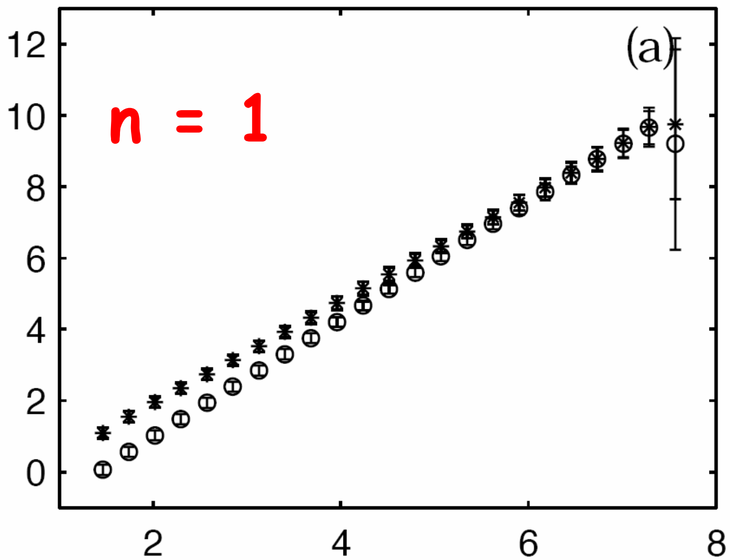


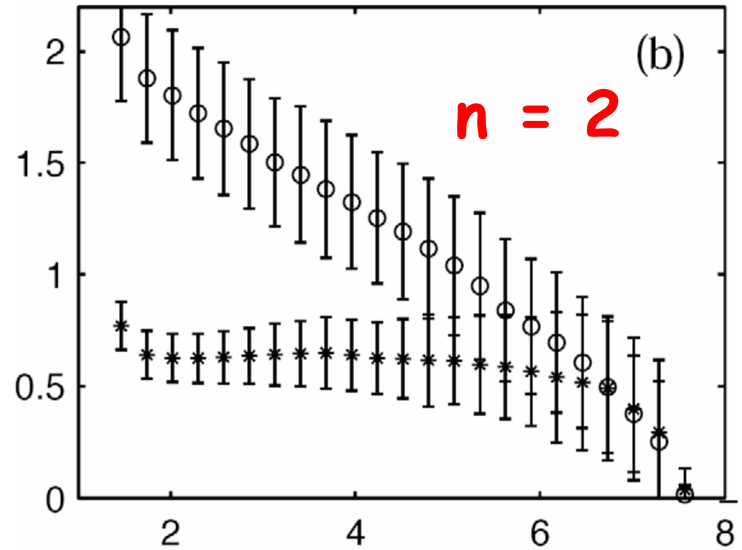
Cannot distinguish FIC from surrogates

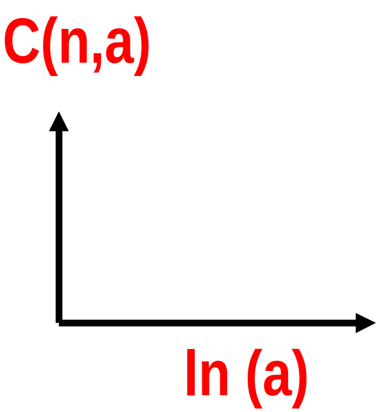
o → Avg. of 100 FICs
* → 100 Surrogates of 100 FICs

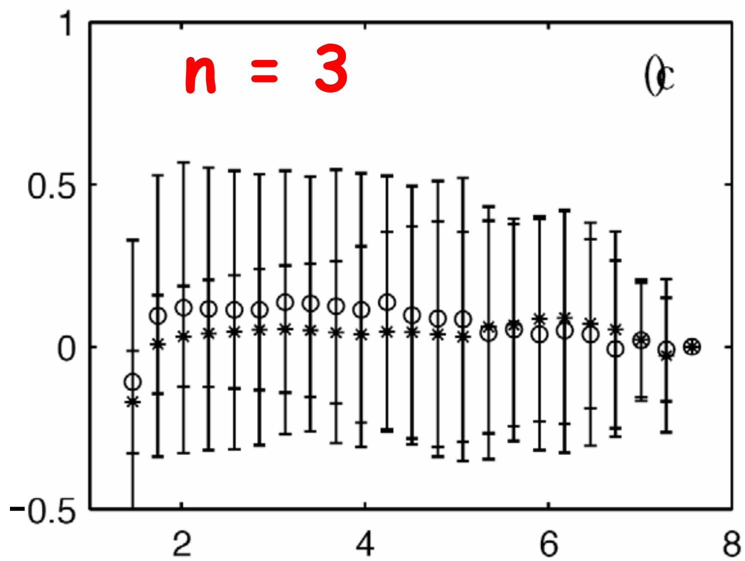
Cumulant analysis of FIC and surrogates

(Ensemble results)









Easy to distinguish FIC from surrogates

o → Avg. of 100 FICs
* → 100 Surrogates of 100 FICs

Bias in estimate of c_1 in surrogates

$$\tau(q) = -c_0 + c_1 q - c_2 \frac{q^2}{2} + \cdots$$

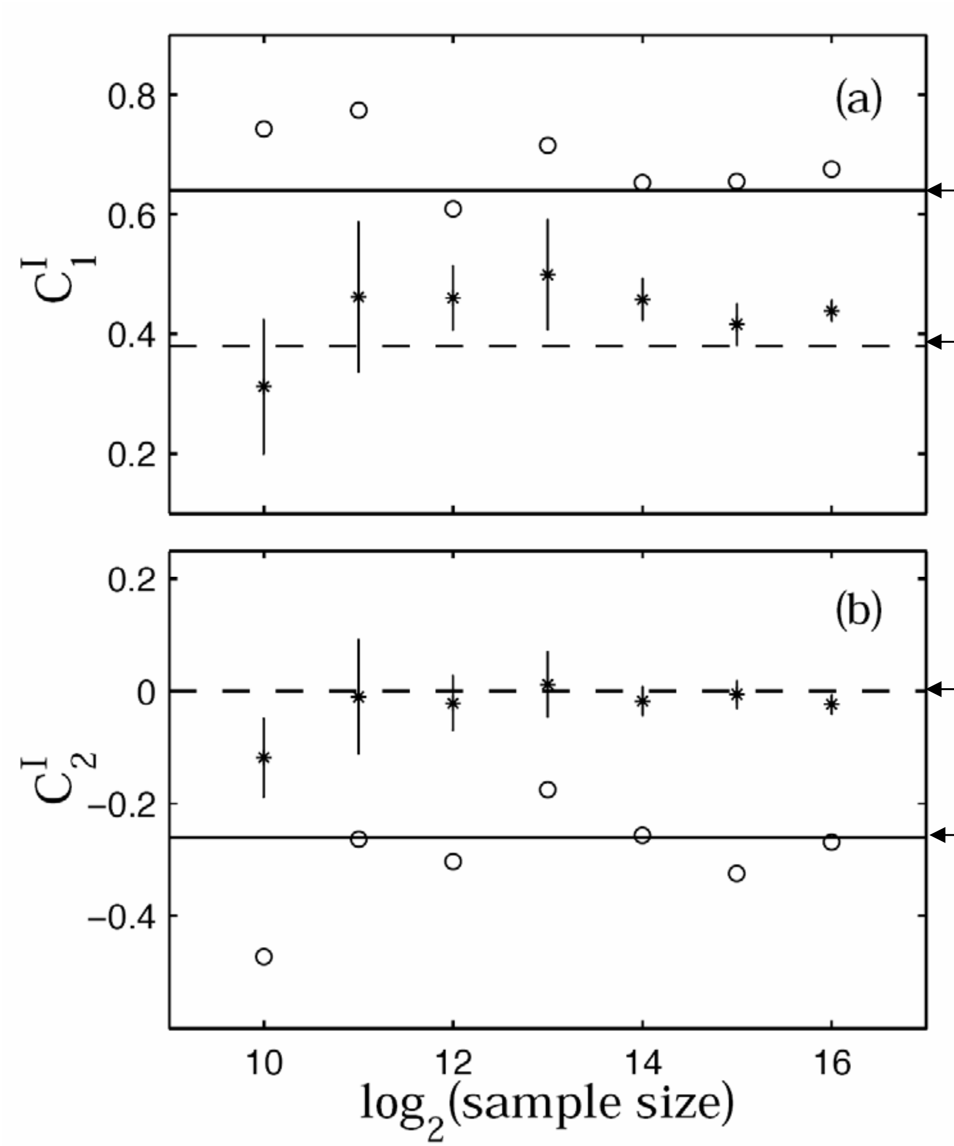
$$\tau(2) = -c_0 + 2c_1 - 2c_2 + \cdots$$

$\tau(2)$ is preserved in the surrogates

FIC ($c_1 = 0.64$; $c_2 = 0.26$) ➔ Surrogates ($c_1' = 0.38$; $c_2' \cong 0$)

Effect of sample size on c_1, c_2 estimates

(FIC vs. Surrogates)



True FIC
(c_1 = 0.64)

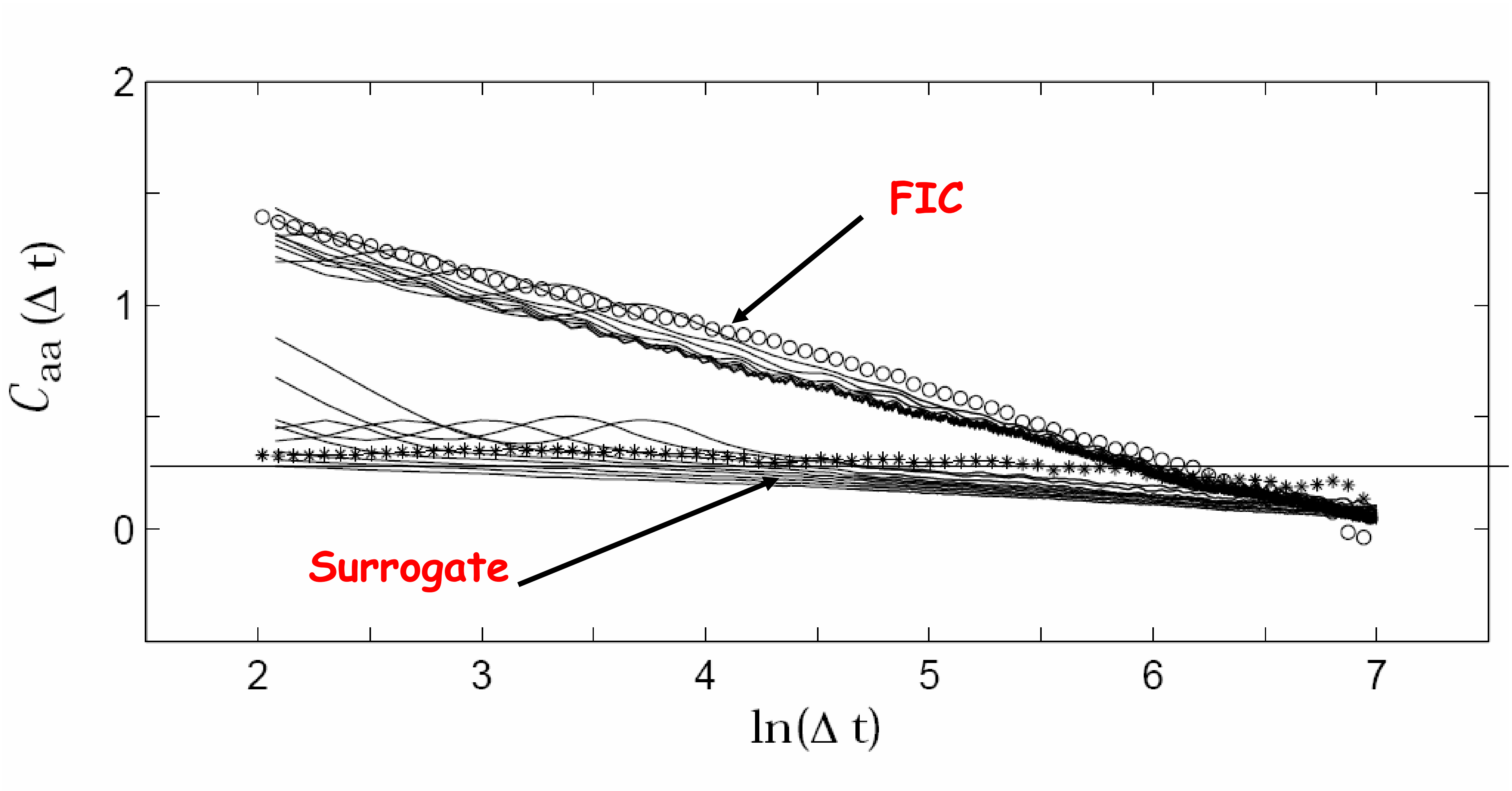
Surrogates
(c_1' = 0.38)

Surrogates
($c_2' \cong 0$)

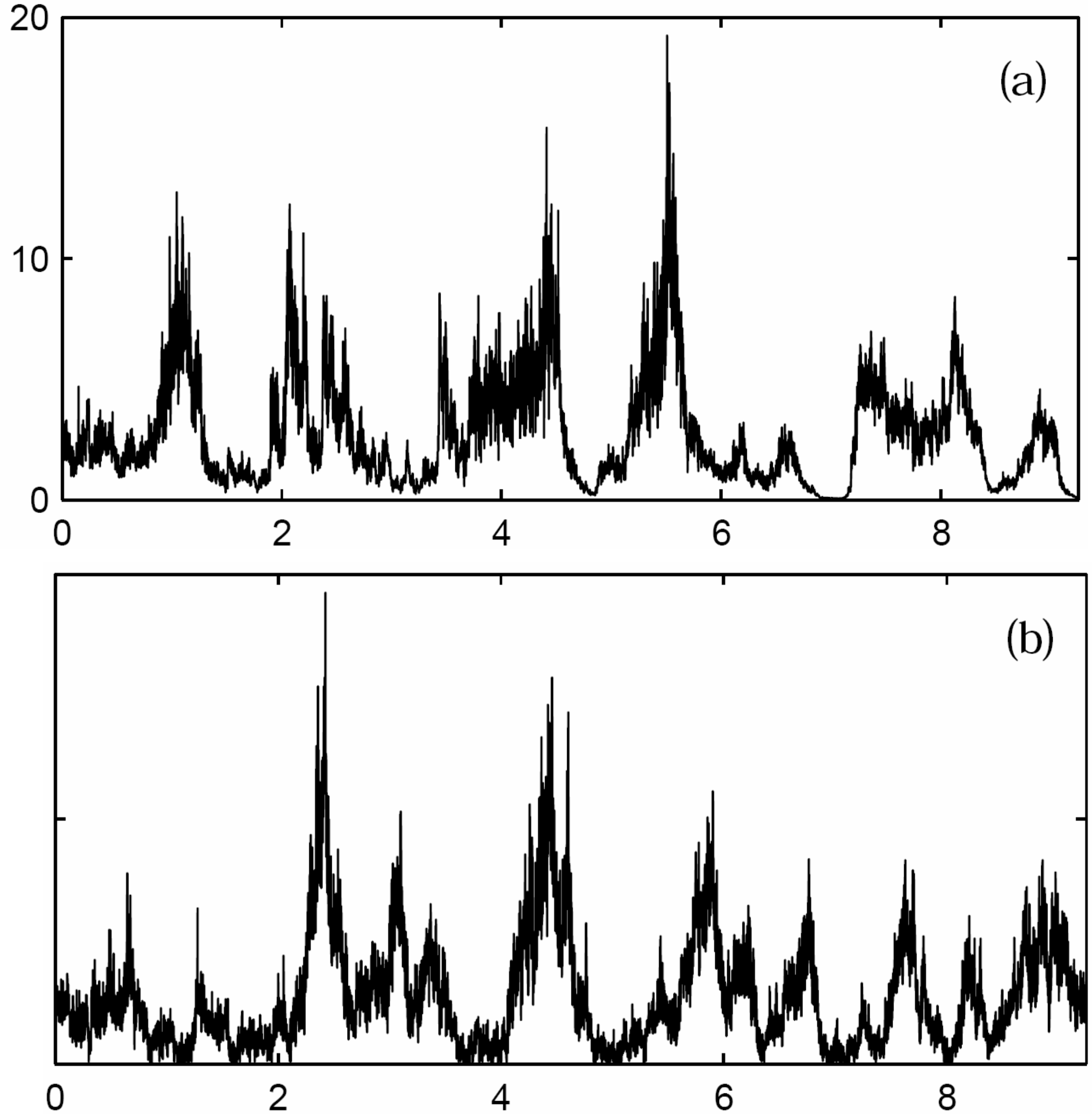
True FIC

o → FIC
* → Surrogates

Two-point magnitude analysis



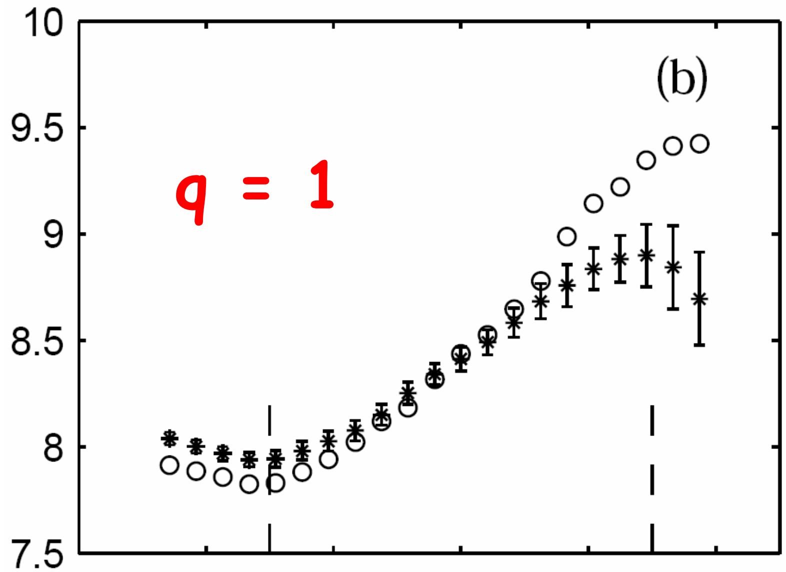
Rainfall vs. Surrogates

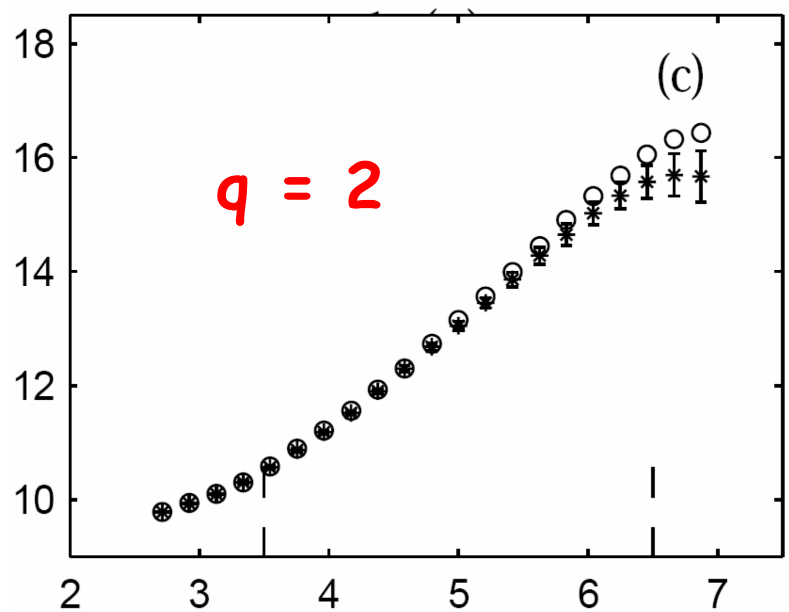


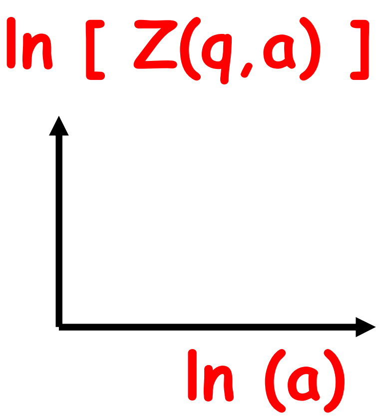
Rainfall

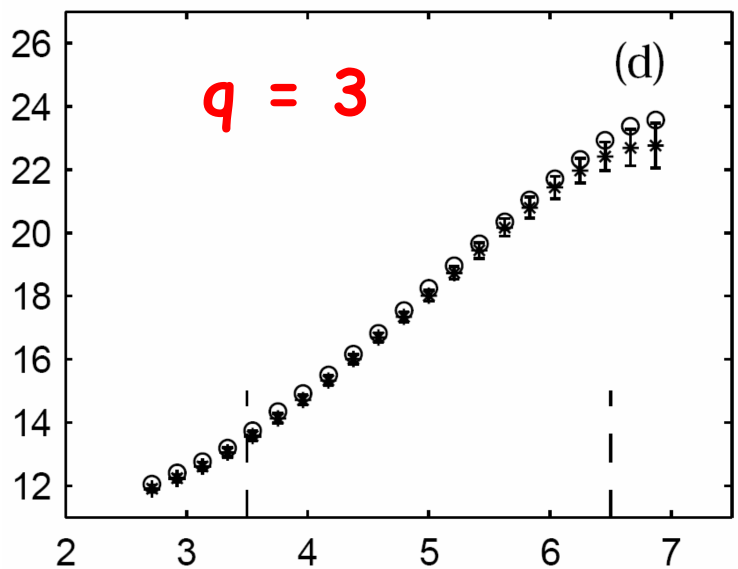
Surrogate

Multifractal analysis of Rain and surrogates





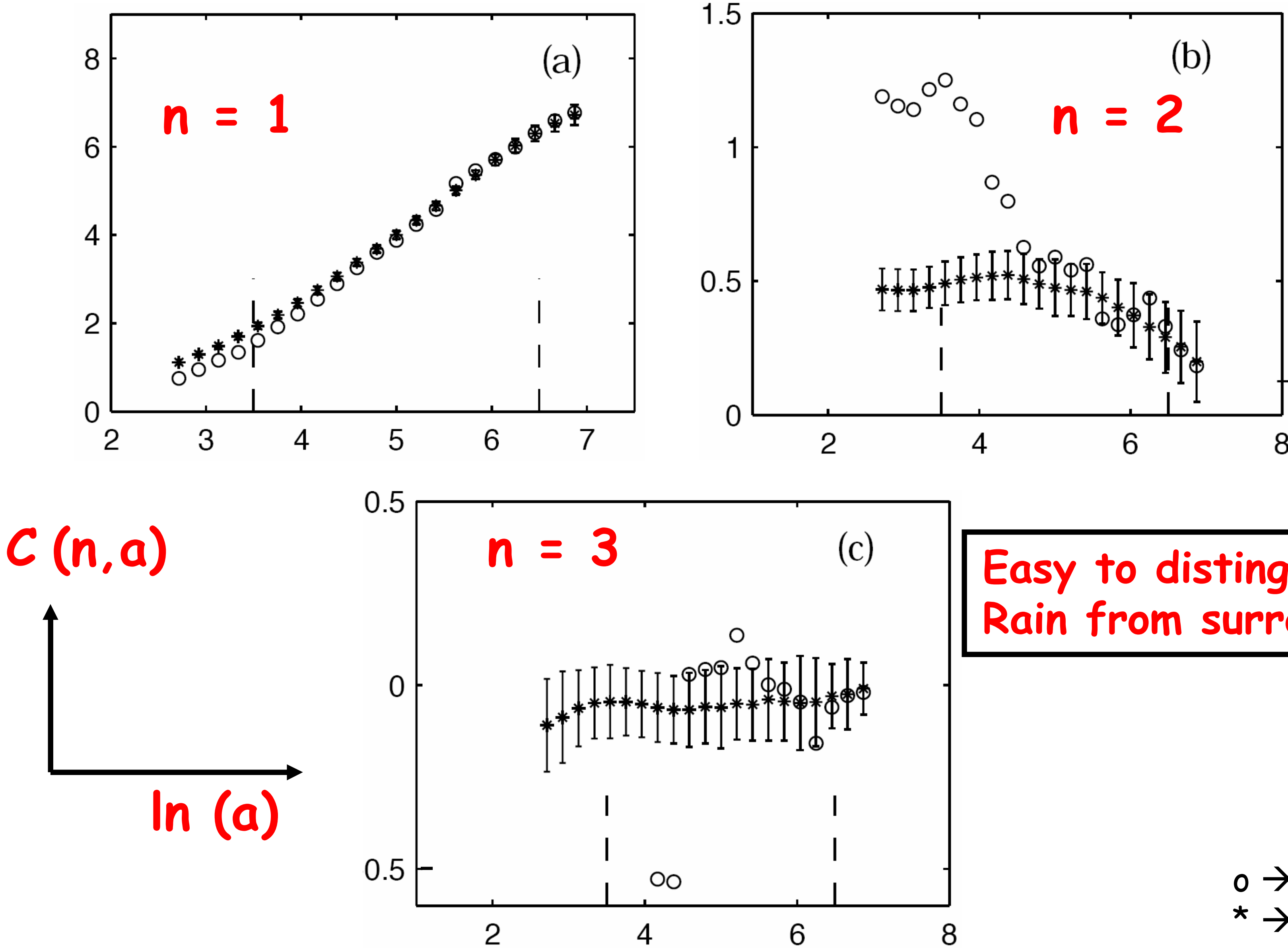




Hard to distinguish
Rain from surrogates

o → Rain
* → Surrogate

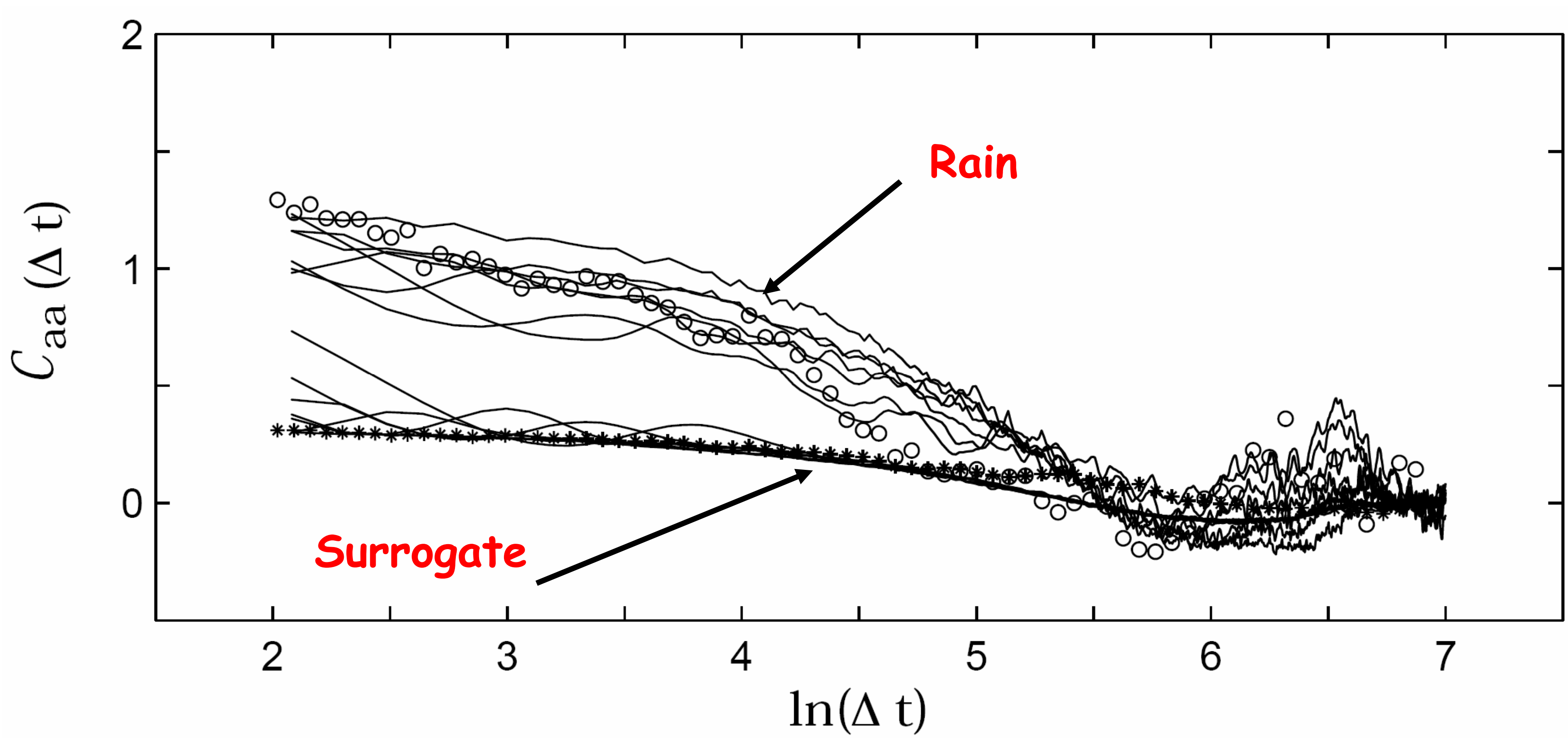
Cumulant analysis of Rain and surrogates



Easy to distinguish Rain from surrogates

o → Rain
* → Surrogate

Two-point magnitude analysis Rain vs. Surrogates



Conclusions

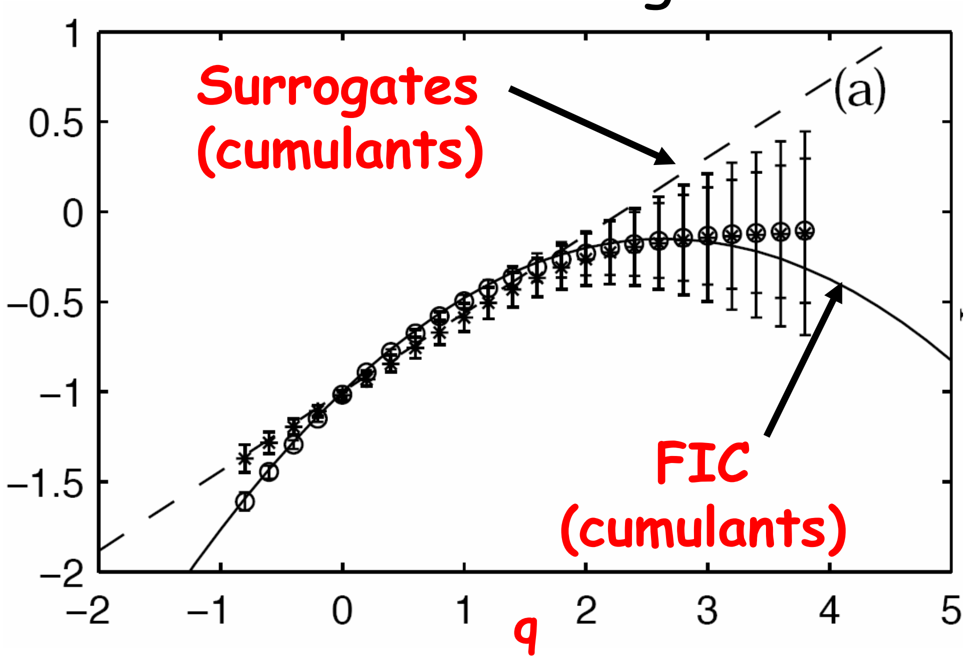
- Surrogates can form a powerful tool to test the presence of multifractality and multiplicativity in a geophysical series
  - Using proper metrics (wavelet-based magnitude correlation analysis) it is easy to distinguish between a pure multiplicative cascade (NL dynamics) and its surrogates (linear dynamics)
  - The simple partition function metrics have low discriminatory power and can result in misleading interpretations
- Temporal rainfall fluctuations exhibit NL dynamical correlations which are consistent with that of a multiplicative cascade and cannot be generated by a NL filter applied on a linear process
- The use of fractionally integrated cascades for modeling multiplicative processes needs to be examined more carefully (e.g., turbulence)

An interesting result…

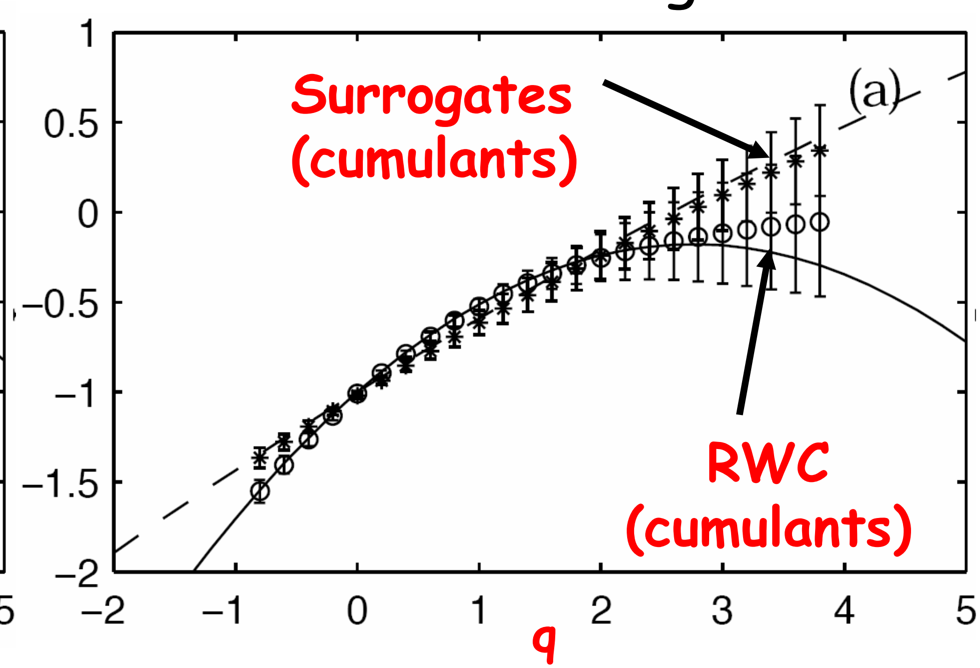
o → FIC
* → Surrogates
(Moments)

o → RWC
* → Surrogates
(Moments)

FIC vs. Surrogates



RWC vs. Surrogates



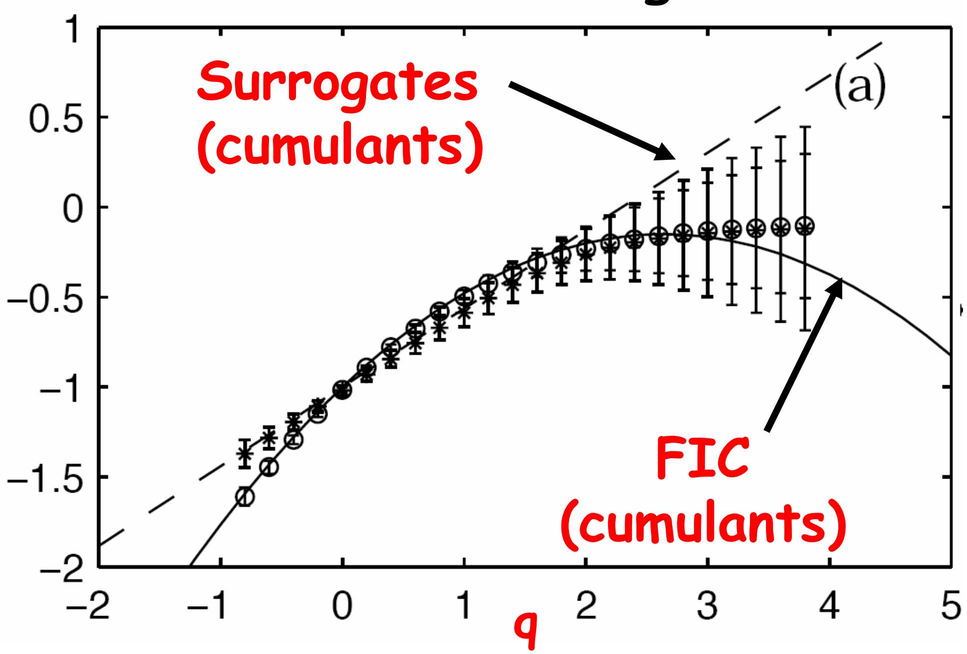
- Surr(FIC): Observed Linear τ(q) for q < 2 and NL for q > 2
- Suggests a "Phase Transition" at q ≅ 2
- τ(q) from cumulants captures behavior at around q = 0 (monofractal)
- Suspect FI operation: preserves multifractality but not the multiplicative dynamics → Test a pure multiplicative cascade (RWC)

An interesting result …

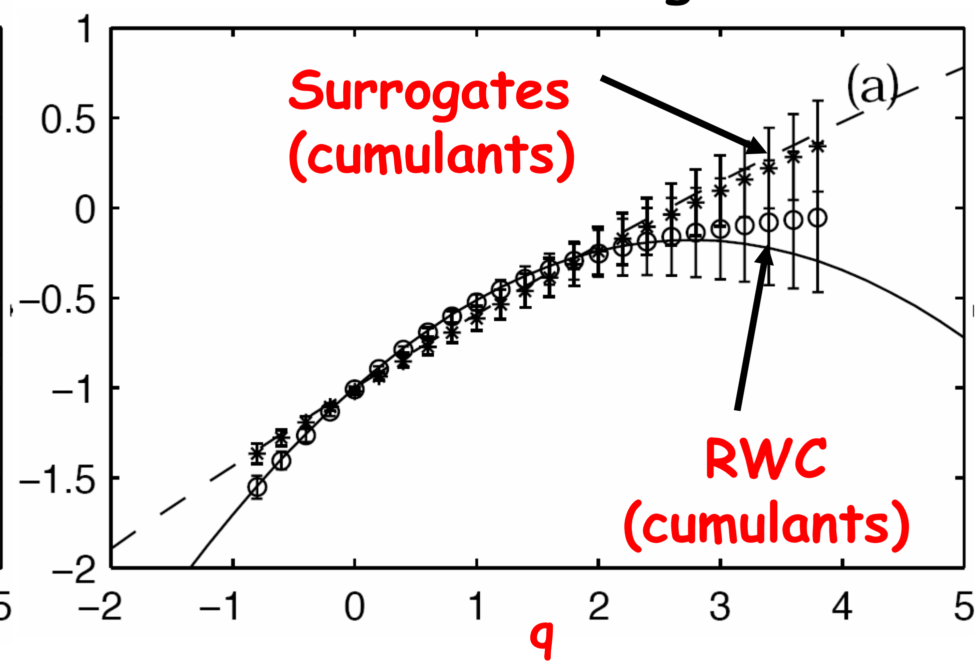
o → FIC
* → Surrogates
(Moments)

o → RWC
* → Surrogates
(Moments)

FIC vs. Surrogates



RWC vs. Surrogates



- IS "Fractionally Integrated Cascade" A GOOD MODEL FOR TURBULENCE OR RAINFALL?

END

Conclusions on Surrogates

- The surrogates of a multifractal/multiplicative function destroy the long-range correlations due to phase randomization
- The surrogates of an FIC show show a "phase transition" at around q=2 (q<2 monofractal, q>2 multifractal). This is because the strongest singularities are not removed by phase randomization.
- The surrogates of a pure multiplicative multifractal process (RWC) show monofractality
- Recall that FIC results from a fractional integration of a multifractal measure and thus itself is not a pure multiplicative process
- Implications of above for modeling turbulence with FIC remain to be studied (surrogates of turbulence show monofractality but surrogates of FIC do not)

Bias in estimate of c_1 in surrogates

$$\tau(q) = -c_0 + c_1 q - c_2 \frac{q^2}{2} + \cdots$$

$$\tau(2) = -c_0 + 2c_1 - 2c_2$$

$$\tau(3) = -c_0 + 3c_1 - \frac{9c_2}{2}$$

FIC: $c_1 = 0.64; c_2 = 0.26$ ➔

$$\tau(2) = -c_0 + 2c_1 - 2c_2 = -1 + 2(0.64 - 0.26) = -0.24$$

$$\tau(3) = -c_0 + 3c_1 - \frac{9c_2}{2} = -1 + 3(0.64) - \frac{9(0.26)}{2} = -0.25$$

Surrogates: c_1', c_2'

$\tau(2)$ is preserved; $c_2' = 0$ ➔

$C_1' = 0.38$

$$\tau(2) = -c_0' + 2(c_1' - c_2') \Rightarrow c_1' = \frac{(\tau(2) + c_0')}{2} + c_2'$$

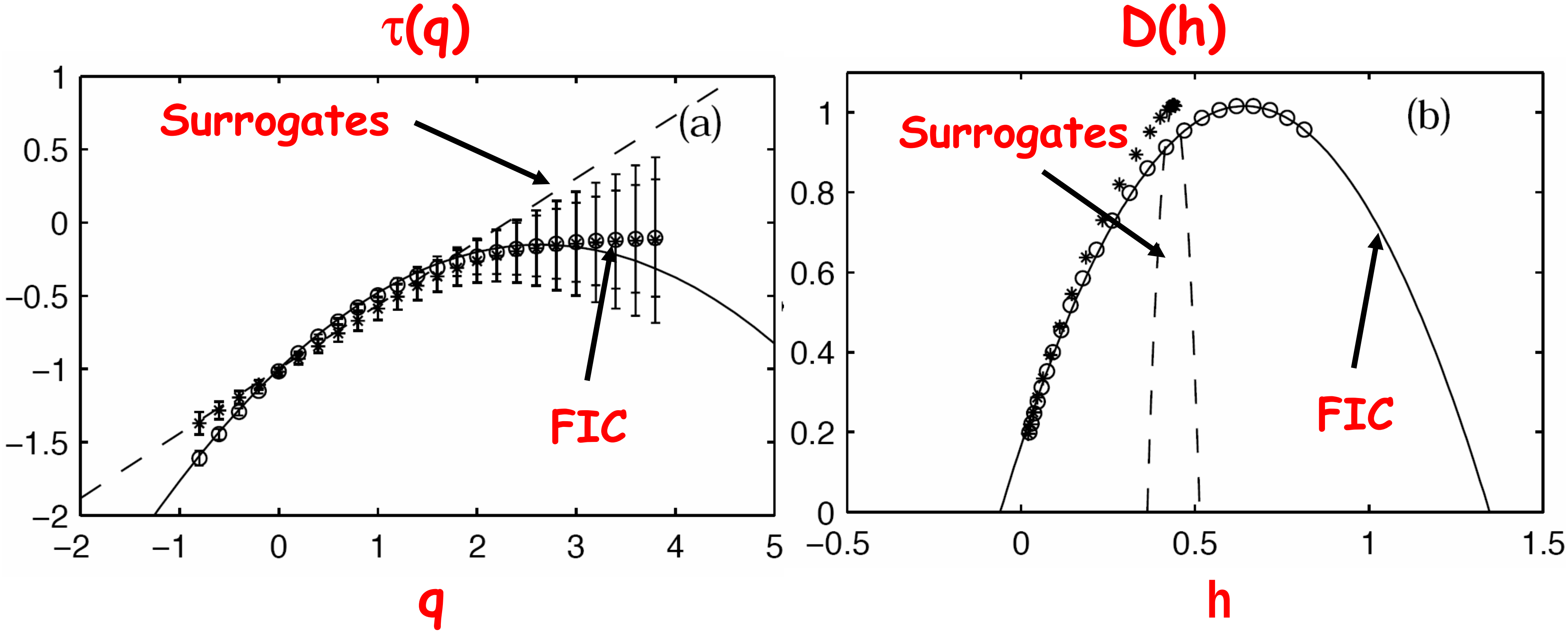
$$\Rightarrow c_1' = 0.38$$

$$\tau(3) = -c_0' + 3c_1' - \frac{9c_2'}{2} = -1 + 3(0.38) - \frac{9(0)}{2} = 0.14$$

Multifractal Spectra: $\tau(q)$ and $D(h)$

(FIC vs. Surrogates)

$c_1 = 0.64$; $c_2 = 0.26$



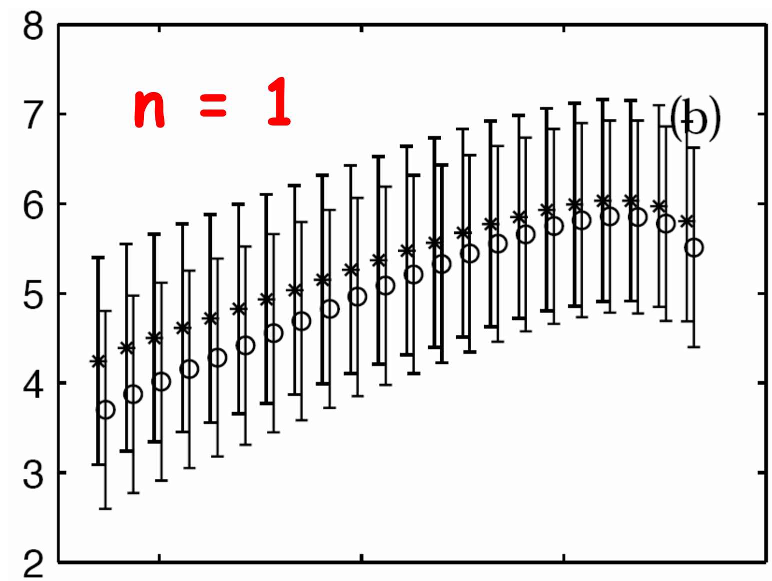
3 slides – RWC vs. Surrogates

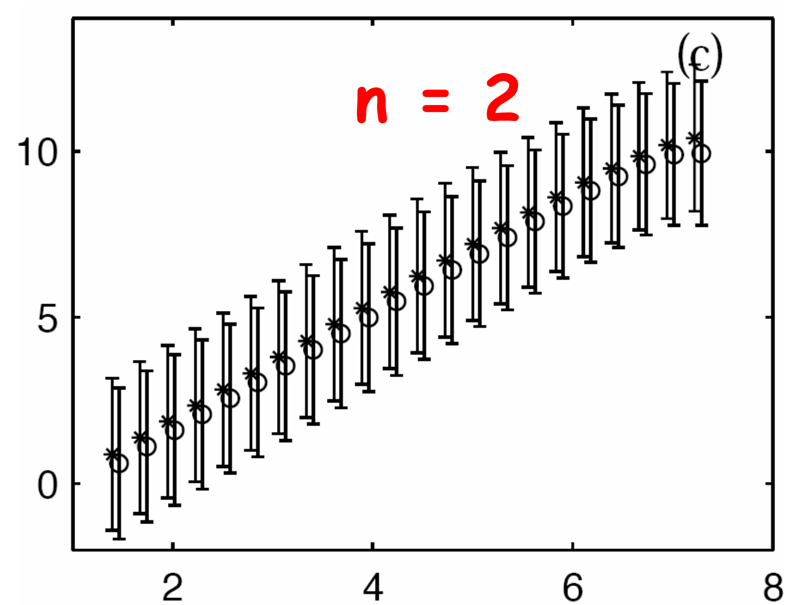
$c_1 = 0.64$; $c_2 = 0.26$

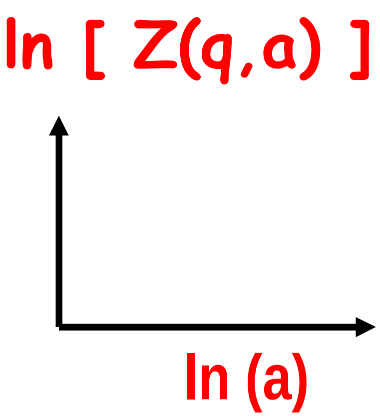
Multifractal analysis of RWC and surrogates

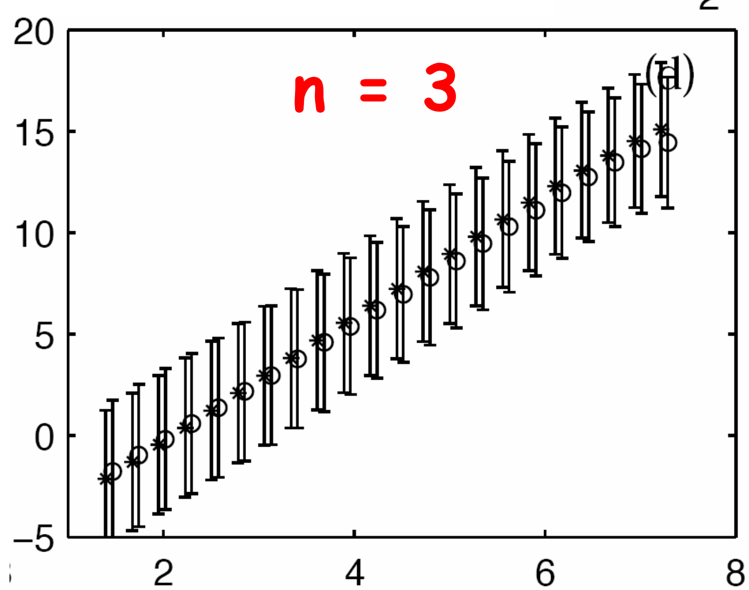
(Ensemble results)

$c_1 = 0.64; c_2 = 0.26$









Cannot distinguish RWC from surrogates

RWC – Random Wavelet Cascade

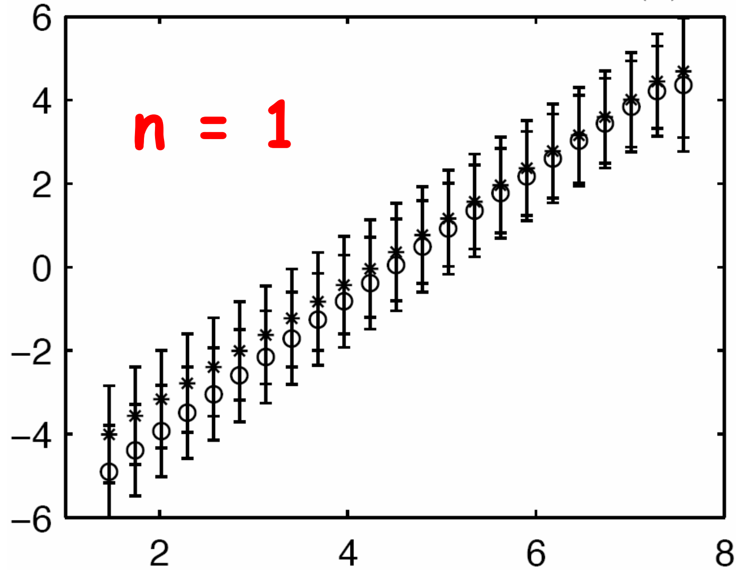
o → Avg. of 100 RWC

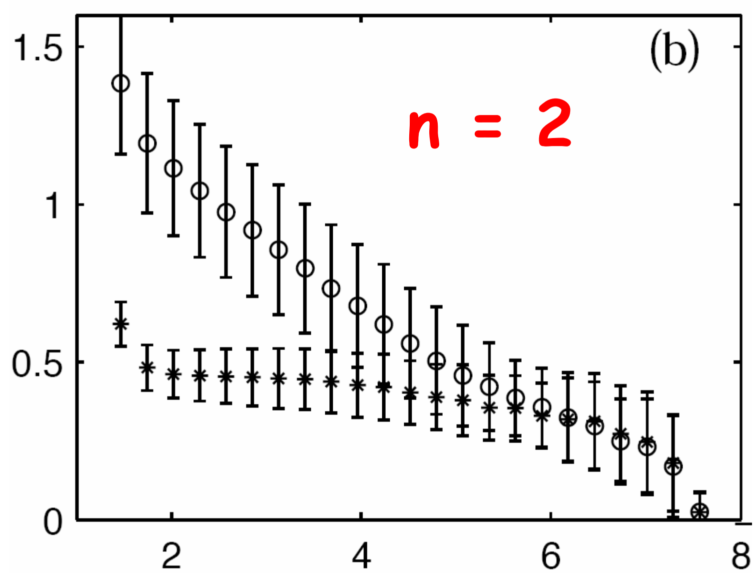
* → 100 Surrogates of 100 RWCs

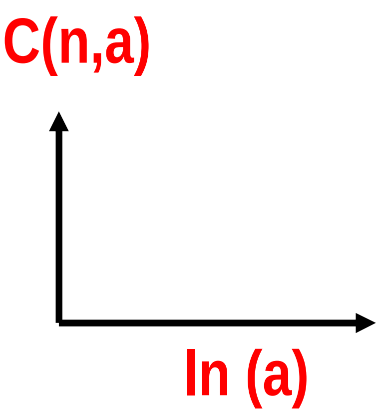
Cumulant analysis of RWC and surrogates

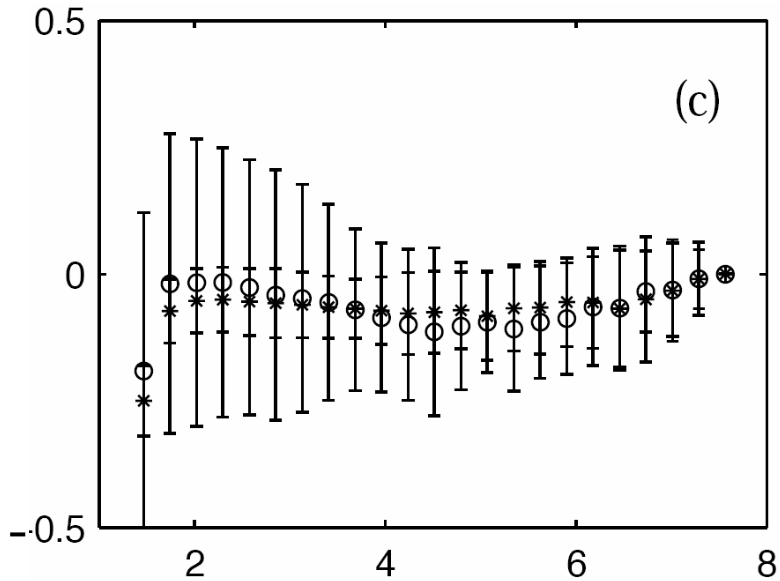
(Ensemble results)

$c_1 = 0.64; c_2 = 0.26$









Easy to distinguish RWC from surrogates

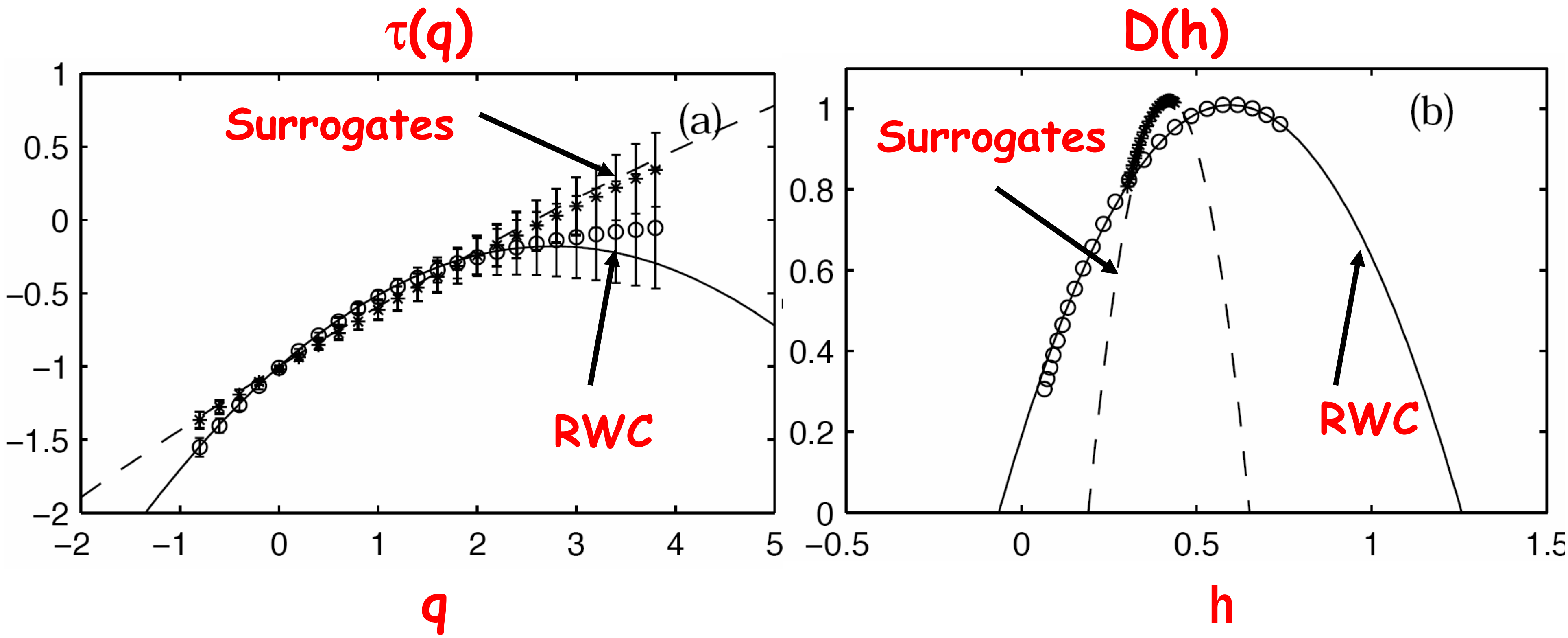
o → Avg. of 100 RWC

* → 100 Surrogates of 100 RWC

Multifractal Spectra: $\tau(q)$ and $D(h)$

(RWC vs. Surrogates)

$c_1 = 0.64$; $c_2 = 0.26$



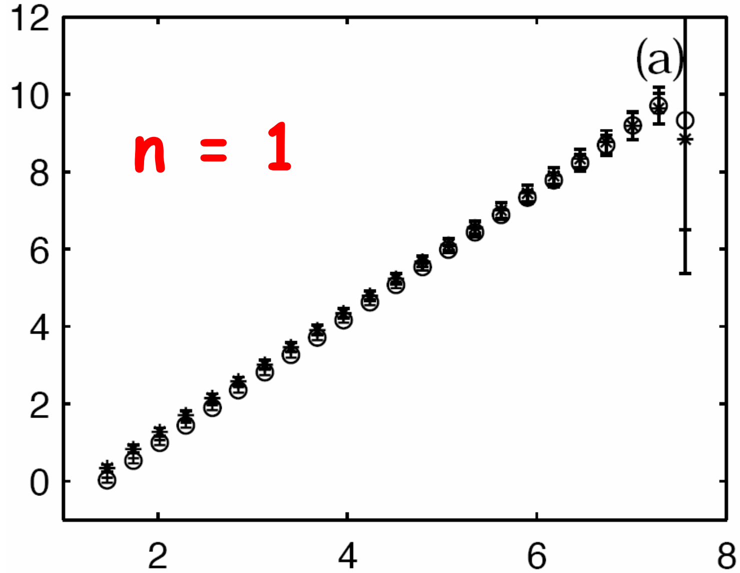
3 slides – FIC vs. Surrogates

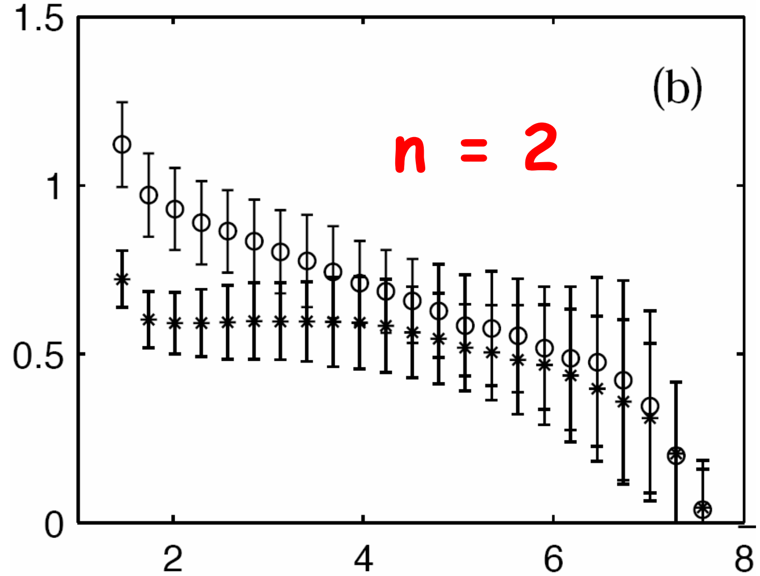
$c_1 = 0.64$; $c_2 = 0.10$

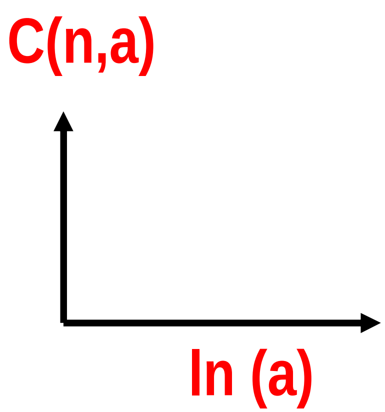
Cumulant analysis of FIC and surrogates

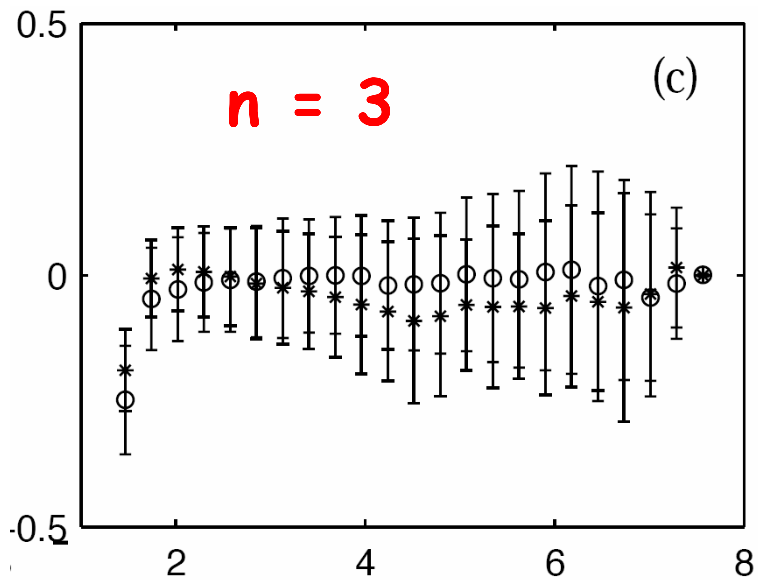
(Ensemble results)

$c_1 = 0.64$; $c_2 = 0.10$









Easy to distinguish FIC from surrogates

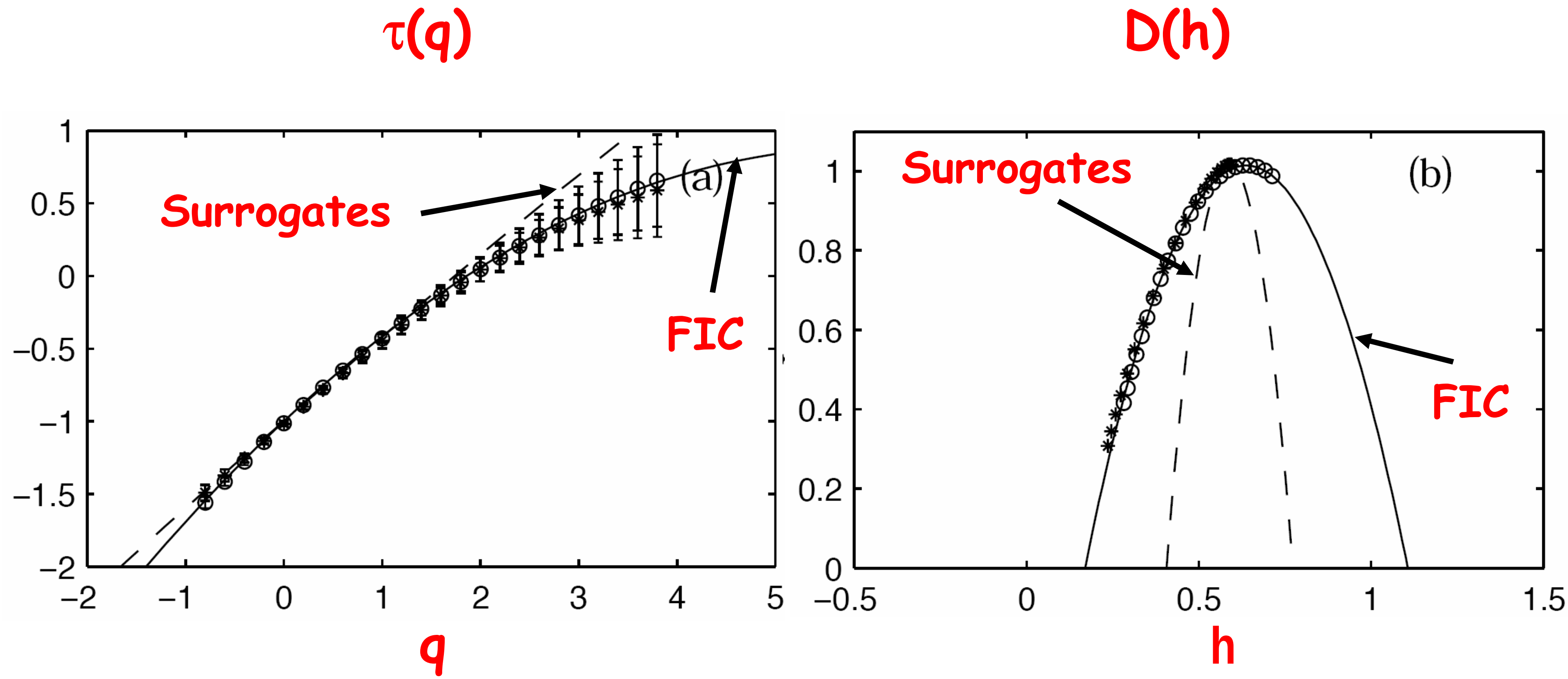
o → Avg. of 100 FICs

* → 100 Surrogates of 100 FICs

Multifractal Spectra: $\tau(q)$ and $D(h)$

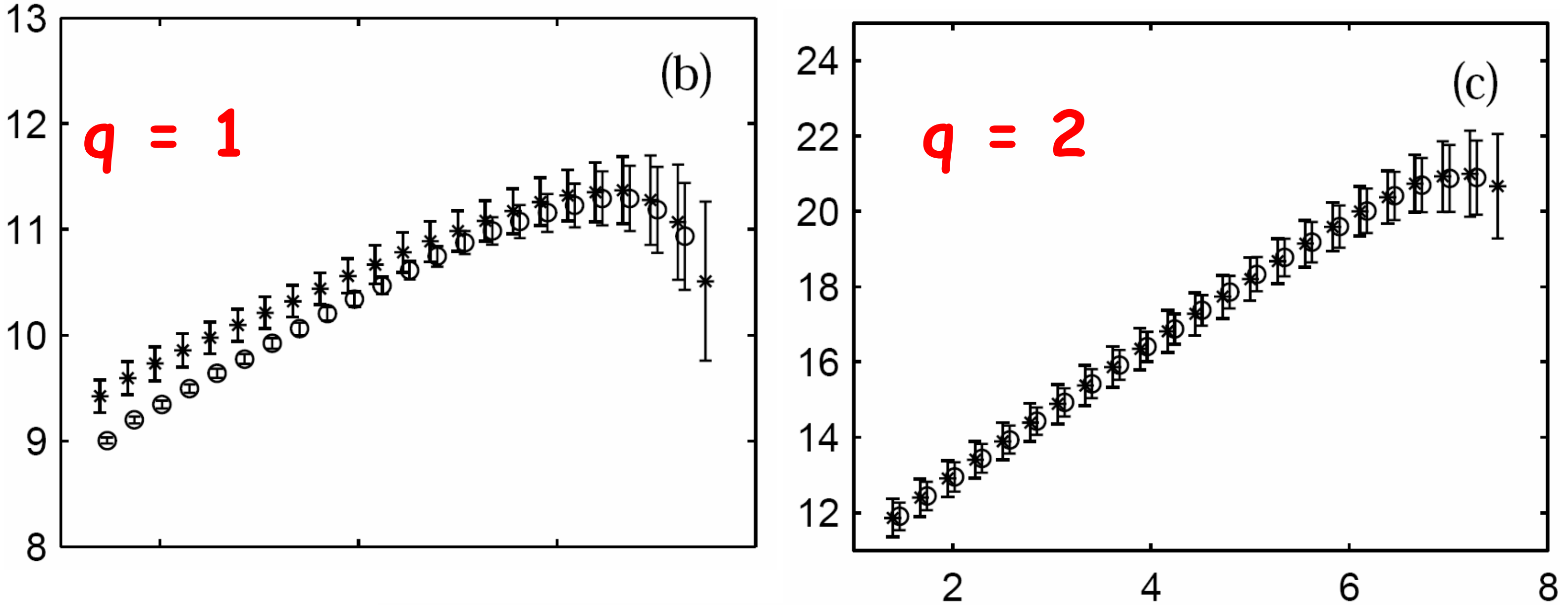
(FIC vs. Surrogates)

$c_1 = 0.64; c_2 = 0.10$



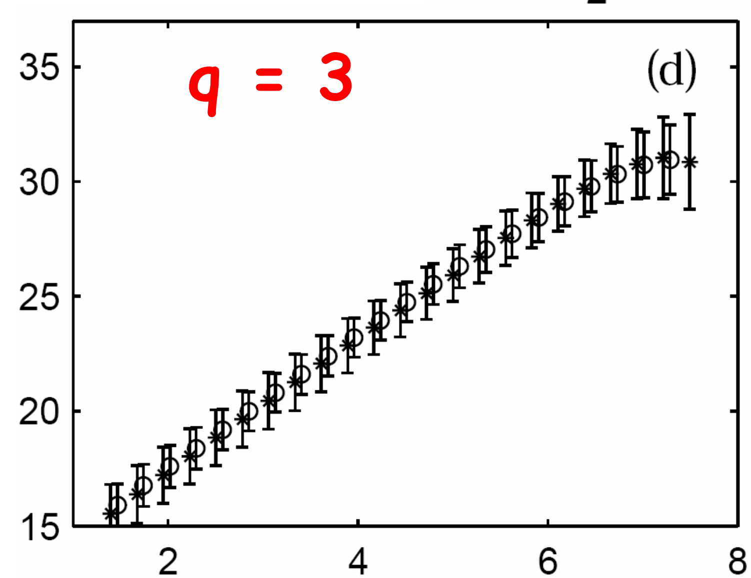
Multifractal analysis of FIC and surrogates

(Ensemble results)



ln [Z(q,a)]

ln (a)



Cannot distinguish FIC from surrogates

o → Avg. of 100 FICs

* → 100 Surrogates of 100 FICs