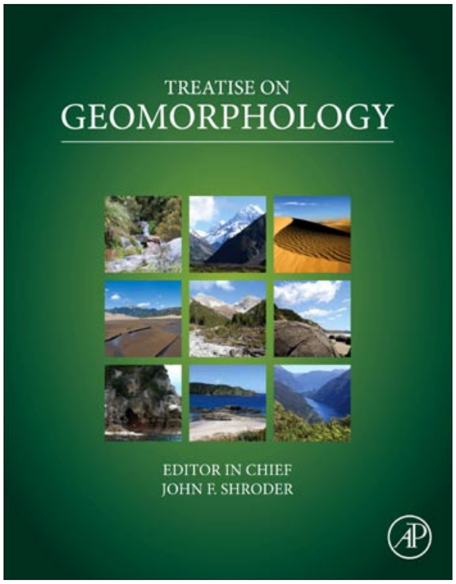

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2.8 Nonlocal Transport Theories in Geomorphology: Mathematical Modeling of Broad Scales of Motion

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Glossary

Anomalous diffusion A term used to describe a diffusion process with a nonlinear relationship to time, in contrast to a typical diffusion process, in which the mean squared displacement of a particle is a linear function of time.

Fokker–Planck equation Describes the time evolution of the probability density function of the velocity of a particle, and can be generalized to other observables as well. It is also known as the Kolmogorov forward equation.

Fractional calculus The study of an extension of derivatives and integrals to noninteger orders.

Geomorphic transport law A mathematical statement, physically motivated but not always derivable from first principles, which expresses the sediment flux or erosion caused by one or more processes on the landscape.

Heavy-tail distributions Probability distributions that possess power law, as opposed to exponential type, decay in the tails. Some statistical moments of these distributions, for example, variance or mean, might not mathematically exist or meaningfully be computed (due to nonconvergence as the sample size increases) from a given set of data.

Nonlocal transport laws Transport laws derived from a flux computation which considers properties of the medium in a neighborhood of the point of interest, that is, a convolution Fickian flux. Nonlocal transport laws capture the broad scales of particle motion as expressed in heavy-tail distributions of waiting times and/or traveled distances.

Abstract

Most geomorphic transport laws proposed to date are local in character, that is, they express material flux at a point (e.g., sediment flux, tracer concentration, and so on) as a function of geomorphic quantities at that point only, such as, elevation gradient, bed shear stress, local entrainment rate, and so on. We present here recent research efforts that argue that nonlocal constitutive laws, in which the flux at a point depends on the conditions in some larger neighborhood around this point in space and/or in time, present a physically motivated alternative to linear and nonlinear diffusion due to their ability to naturally incorporate the presence of heterogeneities known to exist in geomorphic systems. Moreover, this class of models has the potential to eliminate the scale dependence of local nonlinear constitutive laws, which typically require appropriate closure terms. A particularly attractive subclass of these nonlocal constitutive laws involves fractional (noninteger) derivatives in space and/or in time and provides a rich class of models extensively studied in other fields of science. In this chapter, we present examples of nonlocal transport models in a variety of geomorphologic applications, including tracer dispersal in rivers, hillslope sediment transport, and landscape evolution modeling. Nonlocal transport theories is a new and rapidly evolving field of study in the earth sciences and is anticipated to bring new insight into the interpretation of observations and lead to the development of a broader class of models that can explain the stochastic and complex behavior of geomorphic systems over a broad range of space–time scales.

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2.8.1 Introduction

Decades of observations, physical experiments, and numerical simulations in geomorphology have been performed with the goal of understanding the dynamics giving rise to the observed patterns and deriving transport laws able to reproduce those patterns. Problems, such as the spreading of particles downstream of a source in a river, transport of material on hillslopes, the formation of erosional and depositional systems, and contaminant transport in catchments, have been traditionally modeled using simple linear diffusion models (Gilbert, 1909; Culling, 1960, 1963; Hirano, 1968). Linear diffusion models, which assume a linear relation of flux to local gradient (Fick's law), yield at the large time limit (steady-state) patterns that have a specific form. For example, in one-dimensional (1-D) hillslope sediment transport governed by linear diffusion, the expected steady-state hillslope profile is parabolic and, in tracer dispersal governed by a linear diffusion equation, the concentration profile has a Gaussian shape (Einstein, 1908). Ample observations exist (e.g., see discussion in the following sections) that display deviation from these predicted patterns, thus posing questions of generalization and extension of the linear diffusion modeling framework. As will be discussed in the sequel, nonlinear local transport models, in which the local flux depends nonlinearly on the local gradient, have been presented in the literature with success in capturing the observed patterns. Recently however, a new school of thought has emerged in geomorphology, which argues for the limitation of local transport laws (linear or nonlinear) to explicitly handle the large range of scales of transport arising in natural systems and puts forward a new class of models based on the notion of nonlocal flux (referred to as nonlocal transport laws).

The notion of nonlocal flux, as for example, a hillslope sediment flux that depends not only on the local gradient but also on gradients upstream of a point of interest, or a rate of change in tracer concentration at a point that depends on conditions far upstream, challenges the classical view of writing geomorphic transport laws in terms of local attributes. Nonlocality is an expression of the fact that heterogeneities of all scales are present in the system and that the notion of a characteristic length scale over which mass balance is to be performed (or the notion of flux divergence converging as the volume shrinks to zero) loses its meaning.

To understand this concept better, let us start with the well-known advection–dispersion equation (ADE). This formulation is based on the classical definition of divergence of a vector field. The divergence is defined as the ratio of total flux through a closed surface to the volume enclosed by the surface when the volume shrinks to zero (e.g. Schey, 1992; Benson, 1998):

$$\nabla \cdot q_{\mathrm{s}} = \lim_{V \to 0} \frac{1}{V} \iint_{S} q_{\mathrm{s}} \cdot \eta \, \mathrm{d}S \qquad [1]$$

where q_{s} is a vector field, V is an arbitrary volume enclosed by the surface S, and η is a unit normal vector. Implicit in eqn [1] is that the limit of the integral exists, that is, the vector q_{s} exists and is smooth as $V \to 0$.

The classical notion of divergence maintains that as an arbitrary control volume shrinks, the ratio of total surface flux to volume must converge to a single value. However, in many natural systems, variability is known to exist down to very small scales, including abrupt changes and discontinuities. For this reason, a divergence associated with a finite volume and defined as the first derivative of total flux to volume is more relevant. In particular, in systems in which a considerable portion of the mass is contributed from far upstream (e.g., due to a large variability in transport velocities, or due to collective behavior in particle movement), one notes that by increasing the control volume the total surface flux increases nonlinearly (see Figure 1 left panel). Thus, the ratio of total flux to volume does not remain constant but varies with the size of the volume. As a result, the classical diffusion equation is no longer self-contained with a close form solution at all scales.

To adopt the classical theory, the best approximation that can be done is to assume the total flux to volume as piece-wise constant within small ranges of scales (see Figure 1 right panel), allowing one to talk about an effective scale-dependent dispersion coefficient. Alternatively, one can consider mass balance over an infinite volume, or equivalently, consider an integral or convolution Fickian flux (e.g. see Cushman, 1991,

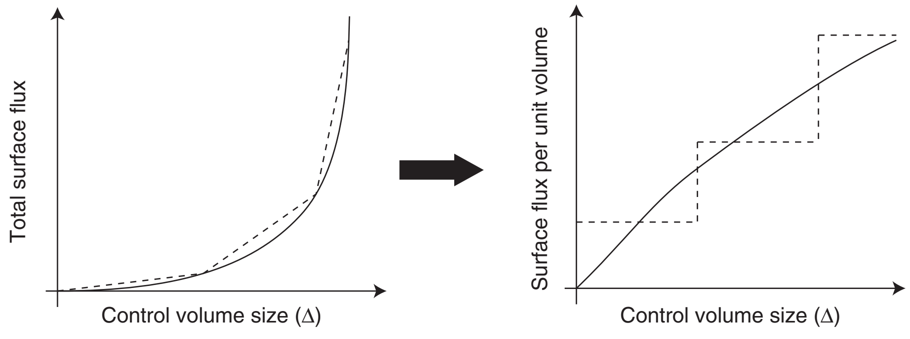


Figure 1 Extended definition of divergence of particle flux when the system exhibits heterogeneity over many scales, as expressed in a nonlinear total flux to volume relationship. Linear approximation of total flux to volume in local neighborhoods (dashed lines in left plot assuming local homogeneity) enables the adoption of classical divergence (constant flux per unit volume shown in the right plot) highlighting, however, the emergence of a scale dependence in the dispersivity coefficient. An extended definition of divergence can be achieved by adopting an integral or convolution Fickian flux using fractional derivatives; see discussion in text. Adapted with permission from Benson, D.A., 1998. The fractional advection–dispersion equation: development and application. PhD dissertation, University of Nevada, Reno.

1997; Benson, 1998). It can be shown that, under the assumption of a convolution kernel that takes a power-law decay form, this flux can be concisely expressed in terms of a fractional (noninteger) derivative (e.g. see Cushman and Ginn, 2000; Schumer et al., 2009).

This chapter reviews recent developments in nonlocal transport laws in geomorphology, with the idea of introducing the reader to this new area of research. In Section 2.8.2, a brief mathematical background on the notion of nonlocal flux and on fractional derivatives is given. In Section 2.8.3, recent work in the context of tracer dispersal in rivers, presenting evidence for deviation of observations from the behavior predicted from classical approaches is discussed and the development of a nonlocal theory able to explain the observed patterns is presented. Section 2.8.4 summarizes recent developments on reformulating theories for sediment transport on hillslopes, whereas Section 2.8.5 discusses the potential of nonlocal theories for landscape evolution modeling. Finally, Section 2.8.6 concludes with future research directions.

2.8.2 Mathematical Background

This section presents the basic mathematical background on the concept of nonlocality and fractional derivatives that provides the foundation for the work presented in this chapter. The reader is referred to Miller and Ross (1993) for an historical survey of fractional calculus and for a detailed exposition of fractional differential equations. A self-contained survey with emphasis on geomorphologic applications was recently presented in Schumer et al. (2009).

In many geomorphic transport systems, the 'flux' can be considered as composed of a collection of particles crossing a given fixed surface in a given amount of time. These particles move downstream, for example, sediment on riverbeds, soil mass on hillslopes, or solutes and contaminants in surface and subsurface flow; their transport is inevitably characterized by the velocities of these traveling particles (or by the distances traveled by the particles in a fixed amount of time) and by the waiting times between particle jumps (i.e., time intervals during which the particles are at rest before they move again). The underlying assumptions of the local transport laws are that the particle travel distances and the waiting times have a thin-tailed distribution whose statistical moments exist and are convergent. As will be discussed in this section, relaxing any of these two assumptions will lead to the notion of nonlocal formulation of flux. Let us first focus on the space component of this motion, that is, the distance traveled by a particle in a given amount of time or, alternatively, the velocities of particles, in the context of tracer transport in rivers. The sediment travel distance is a random variable which is characterized by a probability distribution. The concentration of the tracers at any given spatial location and time, $C(x, t)$, is a surrogate for the distribution of location of tracers in the stream. If the probability distribution of travel distances is thin-tailed (exponential, super-exponential (Gaussian) decay, etc.), that is, the distribution has existing mean and variance, the concentration of tracers spread around the mean location of tracers according to $\sigma = (Dt)^{1/2}$, where D is the coefficient of dispersion (which is a measure of the second statistical moment of the travel distance distribution) and t is the time. This type of scaling is called Fickian or Boltzmann scaling and it implies a local activity, meaning that the particle concentration at a certain location is dependent only on its neighboring locations. In such a case, $C(x, t)$ can be described by a classical ADE.

Let us now consider the case where the particle travel distances have a heavy-tailed distribution, that is, they are characterized by a power-law decay with an exponent $-(\alpha+1)$ where $0<\alpha\leq 2$ is called the tail index. Two distinct cases arise here, namely, when $0<\alpha<1$ and $1<\alpha<2$. In the former case, the particle travel distances do not have an existing first moment (mean) and second moment (variance). However, in the latter case, the particle travel distances have an existing mean, but a nonexisting theoretical second moment (variance). In the case of heavy-tailed travel distances, particles can travel anomalously large distances, albeit with a small (but finite) probability, resulting in a concentration spread around the mean tracer location scaling as $\sigma - (Dt)^{1/\alpha}$ where $1<\alpha<2$. From this scaling relationship, we can notice that the spread grows faster than normal diffusion and thus we call this process 'superdiffusion'. It is also worth noting that the case when $0<\alpha<1$ corresponds to the case which describes a motion which is faster than pure advection ($\alpha = 1$) and is often referred to as 'superadvection'. In such cases, the tracer concentration $C(x, t)$ cannot be modeled using the classical ADE, but one has to move into the realms of generalized transport laws. These include the fractional ADEs with a noninteger order derivative in space (see Benson, 1998; Cushman and Ginn, 2000; Metzler and Klafter, 2000; Meerschaert et al., 1999; Schumer et al., 2009) and the related continuous time random walk (CTRW) models (see Bouchaud and Georges, 1990; Shlesinger et al., 1995; Pekalski and Sznajd-Weron, 1999; Berkowitz et al., 2002).

As discussed earlier, the transport is characterized not only by the particle travel distances but also by their waiting times. In many geomorphic systems, particles can be trapped or stored for long periods of time, for example, when particles are buried in the bed or when particles are stuck in 'dead-zones' which do not participate in the active transport of the system. If the probability density function (pdf) of the waiting times is heavy-tailed with a tail index $\gamma<1$, i.e., the mean of the distribution does not exist, then the spread of the particles scales with time as $t^{\gamma/2}$ with $0<\gamma<1$. In this case, the spread is slower than the one predicted by standard ADE and we call this process 'subdiffusion'. Similar to the case of superdiffusion where the concentration of tracers can be modeled using space-fractional derivatives, subdiffusion can be modeled using time-fractional derivatives where the tail index matches the order of time-differentiation $\frac{d^{\gamma}C}{dt^{\gamma}}$. Note that both superdiffusion and subdiffusion imply that the flux at a given point is not only dependent on the local condition (neighboring locations and present time), but also on the space–time history of the system. The behavior is thus nonlocal either in space or time or both. Note that if both particle distances and waiting times have heavy-tailed distributions, the dispersion scales with a rate proportional to $t^{\gamma/\alpha}$. In practice, a particular value of the rate γ/α estimated from observations, for example, from breakthrough curves of tracer dispersal, can arise from a nonunique combination of values of α and γ. In this case,

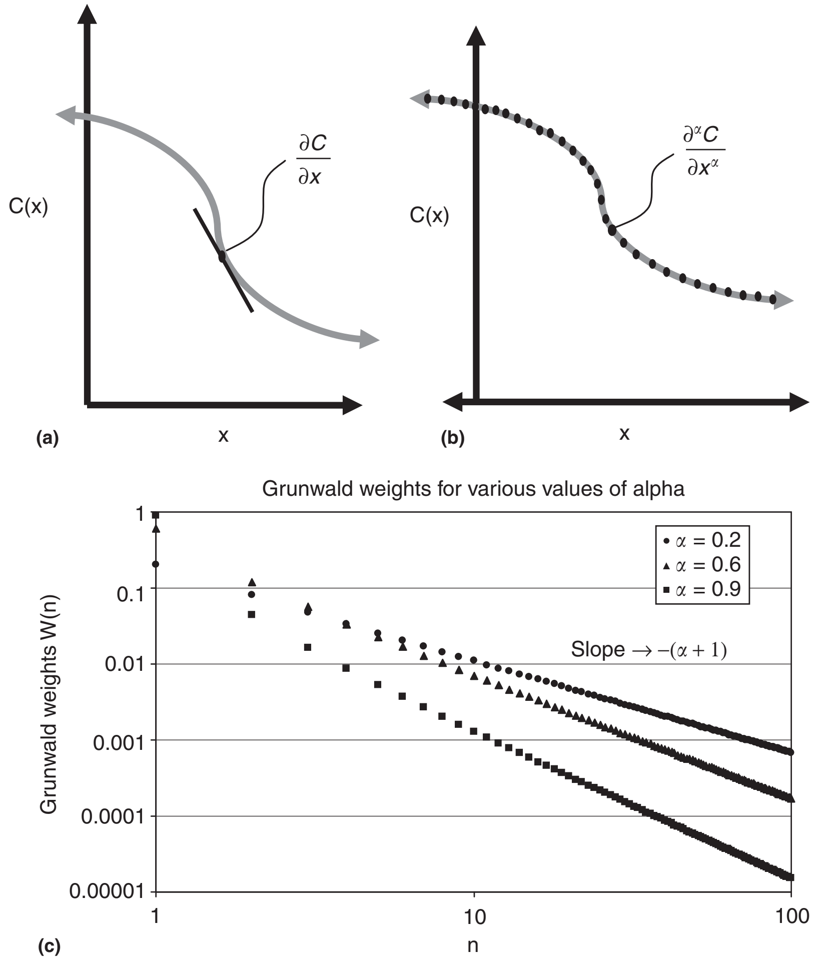


Figure 2 Illustration of the concepts of classical (a) and fractional derivative (b) – notice that while the classical derivative is local, the fractional derivative is nonlocal; (c) shows the power-law decay of the Grunwald weights. Reproduced from Schumer, R., Meerschaert, M.M., Baumer, B., 2009. Fractional advection–dispersion equations for modeling transport at the Earth surface. Journal of Geophysical Research 114, F00A07. doi:10.1029/2008JF001246, with permission from AGU.

additional physical or observational information is needed to differentiate whether this anomalous dispersion has resulted from long particle distances, long waiting times, or both (Schumer et al., 2009).

The concept of nonlocality is connected with the use of fractional derivatives, instead of classical derivatives, in the governing equation. In other words, fractional derivatives are one mathematical means to concisely embed nonlocality in the governing ADE. Fractional derivatives have an interesting history that dates back to 1695 (correspondence between Leibniz, L'Hôpital, and Bernoulli; Laplace, 1820, Fourier, 1822; Lagrange, 1849; see discussion in Miller and Ross, 1993). However, to the best of our knowledge, the first use of fractional operators was made by Abel (1881) and the first text devoted to fractional calculus appeared in 1974 (Oldham and Spanier, 1974). The reader is referred to Miller and Ross (1993) for a thorough exposition and to Sokolov et al. (2002), Metzler and Klafter (2004), and Klafter and Sokolov (2005) for popular expositions of the subject. A recent review is offered by Schumer et al. (2009). Below, we define fractional derivatives and describe how they are computed.

The Grunwald definition of the fractional derivative (Grunwald, 1867) is the noninteger variant of the nth finite difference quotient approximation of the nth-order derivative

$$\frac{d^\alpha}{dx^\alpha} f(x) \approx \frac{1}{h^\alpha} \sum_{j=0}^{\infty} \binom{\alpha}{j} (-1)^j f(x - jh) \qquad [2]$$

with the fractional binomial coefficients given by

$$\binom{\alpha}{j} = \frac{\Gamma(\alpha+1)}{\Gamma(j+1)\Gamma(\alpha-j)} \qquad [3]$$

The approximation in eqn [2] becomes exact as $h \to 0$. Figure 2 shows a representation of the classical and the fractional derivatives, which highlights the nonlocal character of the latter. Note that the Grunwald weights decrease moving far away from the point at which the derivative is computed. Figure 2(c) shows the power-law decay of the Grunwald weights. Table 1 gives examples of fractional derivatives using well-known functions, such as α constant, exponential, and power law.

Table 1 Examples of fractional calculus with $\alpha = \pm 1/2$

Semi-integral	*Function*	*Semi-derivative*
${}_0D_x^{-1/2} f(x) = \frac{d^{-1/2}}{dx^{-1/2}} f(x)$	$f(x)$	${}_0D_x^{1/2} f(x) = \frac{d^{1/2}}{dx^{1/2}} f(x)$
$2C\sqrt{x/\pi}$	C, any const.	$C/\sqrt{\pi x}$
$\sqrt{\pi}$	$1/\sqrt{x}$	0
$x\sqrt{x}/2$	$\sqrt{x}$	$\sqrt{\pi}/2$
$4x^{3/2}/3\sqrt{\pi}$	x	$2\sqrt{x/\pi}$
$\frac{\Gamma(\mu+1)}{\Gamma(\mu+3/2)} x^{\mu+1/2}$	x^{μ}, $\mu > -1$	$\frac{\Gamma(\mu+1)}{\Gamma(\mu+1/2)} x^{\mu-1/2}$
$\exp(x)\mathrm{erf}(\sqrt{x})$	$\exp(x)$	$1/\sqrt{\pi x} + \exp(x)\mathrm{erf}(\sqrt{x})$
$2\sqrt{\pi/x}[\ln(4x) - 2]$	$\ln(x)$	$\ln(4x)/\sqrt{\pi x}$

Source: Adapted from Sokolov, I.M., Klafter, J., Blumen, A., 2002. Fractional kinetics. Physics Today 55, 48–54, with permission from AIP.

A concept intimately related to fractional derivatives is that of 'time subordination'. This concept, roughly speaking, refers to the replacement of standard or clock time with a dynamically changing transformed time. It allows for a nice physical interpretation of fractional derivatives recently presented in Podlubny (2008). We present this interpretation here to provide further insight to the reader.

First, we note that fractional derivatives and fractional integrals relate to each other, that is, a fractional derivative can be seen as the integer-order derivative of a fractionally integrated function (Caputo definition of fractional derivative (Caputo 1967, 1969)):

$$ {}_0D_t^{1-\alpha} f(t) = \frac{d}{dt}\, {}_0I_t^{\alpha} f(t), \quad t \geq 0 \qquad [4] $$

We interpret here the fractional integral and then make an obvious extension to the fractional derivative. The left-sided Riemann–Liouville fractional integral of a function $f(t)$ is defined as (Miller and Ross, 1993)

$$ {}_0I_t^{\alpha} f(t) = \frac{1}{\Gamma(a)} \int_0^t f(\tau)(t-\tau)^{\alpha-1}\, d\tau, \quad t \geq 0 \qquad [5] $$

The above expression can be written as

$$ {}_0I_t^{\alpha} f(t) = \frac{1}{\Gamma(a)} \int_0^t f(\tau)\, dg_t(\tau) \qquad [6] $$

with

$$ g_t(\tau) = \frac{1}{\Gamma(\alpha)} \{t^{\alpha} - (t-\tau)^{\alpha}\} \qquad [7] $$

If we see τ as clock time in eqn [5], $g_t(\tau)$ can be seen as a 'transformed timescale' given by eqn [7]. In other words, eqn [7] can be seen as transforming a homogeneous time axis (expressed by τ) to an inhomogeneous or dynamically changing time axis (expressed by $g_t(\tau)$). As a result, the fractional integral of eqn [5] over clock time τ is equivalent to a standard integral over a transformed time $g_t(\tau)$. We call the function $g_t(\tau)$ a subordinator, which "stretches" time, that is, time passes faster for a fast-moving particle, or slower for a particle that is trapped. This development highlights the appealing physical interpretation of fractional derivatives in terms of artificially creating (by shrinking or expanding time) a 'scale separation' in, rendering thus the ordinary differential operator (and the classical divergence) valid in this transformed time domain. It is noted that the function $g_t(\tau)$ has an interesting scaling property, namely $g_{kt}(k\tau) = k^{\alpha} g_t(\tau)$. Notice that for $\alpha = 1$, $g_t(\tau) = \tau/\Gamma(\alpha+1)$, that is, it is independent of t, retrieving the standard integer-order integral of $f(t)$. However, for $\alpha \neq 1$ (i.e. dynamically changing time axis), $g_t(\tau)$ depends both on t and τ.

2.8.3 Superdiffusion in Tracer Dispersal

Quantifying the dispersal of tracers in gravel bed rivers not only is important for accurately modeling pollutant transport, but also forms the foundation of probabilistic models for bedload transport computation. The process of tracer dispersal can be seen as a stochastic process, recognizing the randomness of motion of individual particles.

We start by introducing the 1-D Exner equation for sediment balance (Tsujimoto, 1978; Parker et al., 2000; Garcia, 2008):

$$ (1-\lambda_p)\frac{\partial \eta(x,t)}{\partial t} = D_b(x,t) - E_b(x,t) \qquad [8] $$

where η denotes the local mean bed elevation, t denotes time, x is the downstream coordinate, D_b is the volume rate per unit area of deposition of bedload particles onto the bed, E_b denotes the volume rate per unit area of entrainment of bed particles into bed load, and λ_p is the porosity of bed sediment. First, we assume that a particle, once entrained, undergoes a step of length r before redepositing. Then, assuming that this step length is a random variable with probability density $f_s(r)$ the deposition rate of tracers is given by

$$ D_b(x,t) = \int_0^{\infty} E_b(x-r,t) f_s(r)\, dr \qquad [9] $$

The above formulation takes into account the effect of stochasticity on step length, but not on resting time. Making now the assumption of an active layer of thickness L_a, grains within it exchange directly with bedload grains, whereas grains below the active layer exchange only by bed aggradation and degradation. Thus, the equation of mass conservation of tracers can be simplified as follows:

$$ (1-\lambda_p)\left(f_I(x,t)\frac{\partial \eta(x,t)}{\partial t} + L_a \frac{\partial f_a(x,t)}{\partial t}\right) = D_{bT}(x,t) - E_{bT}(x,t) \qquad [10] $$

where $f_a(x,t)$ indicates the fraction of tracer particles in the active layer at location x and time t, $f_I(x,t)$ is the fraction of tracer particles exchanged at the interface between the active layer and the substrate, E_{bT} is the volume entrainment rate of tracers, and D_{bT} is the corresponding deposition rate. E_{bT} and D_{bT} are given by the following expressions

(Parker et al., 2000)

$$E_{\mathrm{bT}}(x,t) = E_{\mathrm{b}}(x,t) f_{\mathrm{a}}(x,t) \qquad [11]$$

$$D_{\mathrm{bT}}(x,t) = \int_0^\infty E_{\mathrm{b}}(x-r,t) f_{\mathrm{a}}(x-r,t) f_{\mathrm{s}}(r)\,\mathrm{d}r \qquad [12]$$

Under the assumption of bed elevation at equilibrium, L_{a}, η, and $f_{\mathrm{s}}(r)$ are constant in space and time and eqns [10]–[12] reduce to:

$$(1-\lambda_{\mathrm{p}}) \frac{L_{\mathrm{a}}}{E_{\mathrm{b}}} \frac{\partial f_{\mathrm{a}}(x,t)}{\partial t} = \int_0^\infty f_{\mathrm{a}}(x-r,t) f_{\mathrm{s}}(r)\,\mathrm{d}r - f_{\mathrm{a}}(x,t) \qquad [13]$$

It can be shown (Ganti et al., 2010) that in the case in which $f_{\mathrm{s}}(r)$ in eqn [9] is thin tailed, the above formulation leads at the large time limit to the classical ADE:

$$\frac{L_{\mathrm{a}}}{E_{\mathrm{b}}} \frac{\partial f_{\mathrm{a}}}{\partial t} = -v \frac{\partial f_{\mathrm{a}}}{\partial x} + D_{\mathrm{d}} \frac{\partial^2 f_{\mathrm{a}}}{\partial x^2} \qquad [14]$$

where $v=\mu_1$ and $2D_{\mathrm{d}}=\mu_2$ and μ_1 and μ_2 are the first- and the second-order moments of the step length distribution. In the case of a pulse of tracers as initial condition at $t=0$, we recover the Green's function of the above equation, namely a Gaussian distribution of the tracer concentration at any given time $t>0$, with variance directly related to t. In the case of a distributed source of tracers in space and/or in time (as opposed to a pulse), the solution is given by the convolution of the Green's function with the source.

We would like to discuss now whether or not eqn [14] is always able to reproduce the patterns observed in river transport. What the classical ADE implies is that tracers spread downstream with a constant diffusivity. There are cases, though, in which particles undergo very fast velocities and unusually long retardation due to trapping and the resulting transport patterns are not characterized anymore by the standard ADE, as local diffusivities vary greatly and are not well described by an average value or by a single, effective, diffusivity parameter. Evidence is given, for example, by the data of the tracer experiment performed by Sayre and Hubbell (1965). This experiment involved the release of radioactive tracers (sand plated with iridium-192) in the North Loup River in Nebraska and monitoring the fate of this radioactive material (via gamma ray count rate) during a period of 13 days over a 548.8-m stretch of the river downstream of the release point. The data showed that a high fraction of tracers was in the downstream tail of the distribution, the detected tracer mass was decreasing over the course of the experiment, and there was enhanced particle retention near the source (see Figure 3).

Obviously, such an activity cannot be explained by the classical ADE, and Sayre and Hubbell (1965) attempted a modified model that improved upon the earlier predictions without, however, being able to reproduce all aspects of the data. Recently, new models based on fractional ADEs have been proposed with the potential of better capturing the observed patterns. These models are explained later.

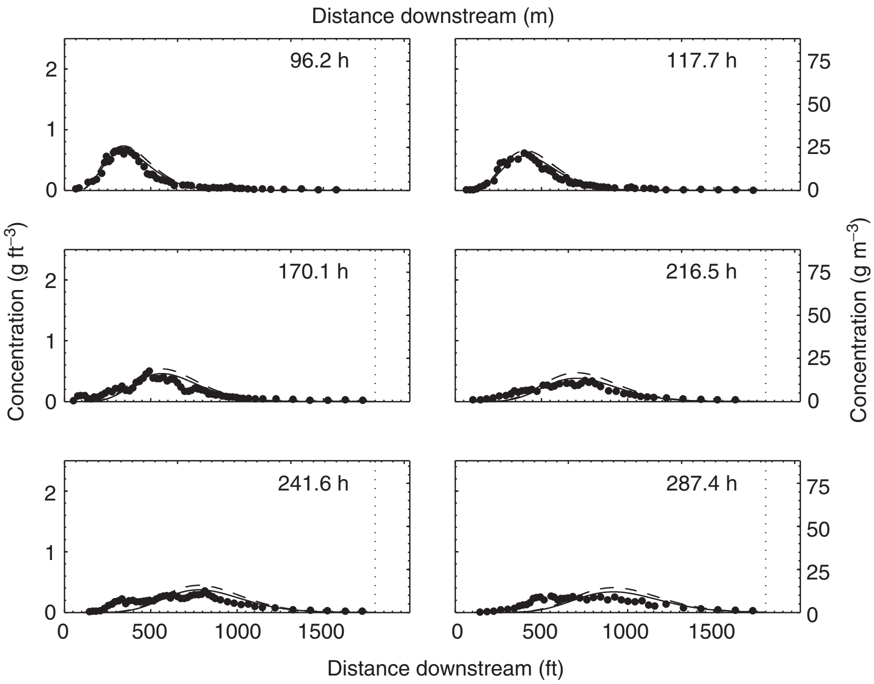


Figure 3 Sayre and Hubbell (1965) experimental data. The panels show the time evolution of the tracer plume. Data are shown from traverses along the right side of the channel on 10 different days. The black dotted lines indicate the estimated concentration from the experiment, whereas the solid and dashed black lines indicate the predicted concentration through a classical advection–dispersion model, and under two different rescaling formulations. Concentrations represent the tracer mass divided by channel width and mixing depth. As it can be seen, the classical advection–dispersion equation (ADE) does not explain the observed concentration patterns. Adapted from Bradley, D.N., Tucker, G.E., Benson, D.A., 2010. Fractional dispersion in a sand bed river. Journal of Geophysical Research 115, F00A09. doi:10.1029/2009JF001268, with permission from AGU.

Ganti et al. (2010) reformulated the probabilistic Exner equation by considering that the probability distribution of particle displacement has a heavy tail, that is, very long displacements are often possible albeit with small probability. In this case, the step-length distribution is given by

$$f_s(r) \approx C\,\alpha r^{-\alpha-1} \quad [15]$$

where $r>0$, C is a positive constant, and $1<\alpha<2$ is the power–law index. Ganti et al. (2010) derived the long-time limit, continuum constitutive equation in this case and showed that it takes the form of a fractional ADE:

$$\frac{L_a}{E_b}\frac{\partial f_a}{\partial t} = -v\frac{\partial f_a}{\partial x} + D_d\frac{\partial^\alpha f_a}{\partial x^\alpha} \quad [16]$$

including, as a special case, the standard ADE when the tails of the particle displacement probability distribution can be approximated by exponential-type decay. They argued that the heavy-tail probability distribution of particle step lengths can easily arise in nature from the convolution of an exponential probability distribution of displacements, conditional on a specific particle size, with a broad probability distribution of particle sizes (see also Hill et al. (2010) for evidence of such a heavy-tailed distribution).

Figure 4 shows the long-time asymptotic solutions of the anomalous ADE for three different values of α, namely $\alpha = 1.1$, 1.5, and 2, where the last value corresponds to normal ADE (Gaussian distribution of tracer concentration). Note that the parameter α, being related to the heaviness of the tail of the probability distribution of particle step lengths, indicates how far downstream the particle will disperse from the initial location. As can been seen in Figure 4, after 500 days, only $\sim$5% of the tracers are recovered at 550 m downstream of the release point according to the classical ADE model, whereas $\sim$8% and $\sim$18% are recovered according to the fractional ADE with $\alpha = 1.5$ and $\alpha = 1.1$, respectively (superdiffusive behavior).

Bradley et al. (2010) proposed a fractional advection–dispersion model and a two-phase transport formulation to model tracer dispersal in rivers and explained both the leading tail and the enhanced retention near the source, documented in the observations of Sayre and Hubbell. Writing the fractional ADE in the simplest form

$$\frac{\partial C}{\partial t} = -v\frac{\partial C}{\partial x} + D\frac{\partial^\alpha C}{\partial x^\alpha} \quad [17]$$

the random walk solution proposed by Bradley et al. (2010) is a particle-tracking model that includes five parameters: the mean mobile velocity $\bar{u}_m$, the dispersion coefficient D, the mean τ_m of the exponential distribution of the particle flight time, the mean τ_{im} of the exponential distribution of the particle resting time, and the tail index α. The results obtained by Bradley et al. (2010) after an empirical fitting of the five parameters showed a better agreement of the theoretical prediction to the experimental observations than with the classical advection–dispersion formulation, but slightly overpredicted the peak concentration (see Figure 5). By assuming a mobile/immobile model (MIM) along the lines of that proposed by Harvey and Gorelick (2000), Bradley et al. (2010) obtained a constraint on the time parameters τ_m and τ_{im} and the means to scale the model prediction to an absolute concentration in order to account for the enhanced retention near the source and the decrease in the detected mass observed during the course of the experiment. The concentration profiles obtained with this model (see Figure 6) show an impressive agreement with the Sayre and Hubbell experimental data, indicating that, although a simple fractional ADE can reproduce the heavy leading edge of a tracer plume, the distinction between detectable (mobile) and undetectable (immobile) particles is needed to reproduce the observed decrease in detected mass.

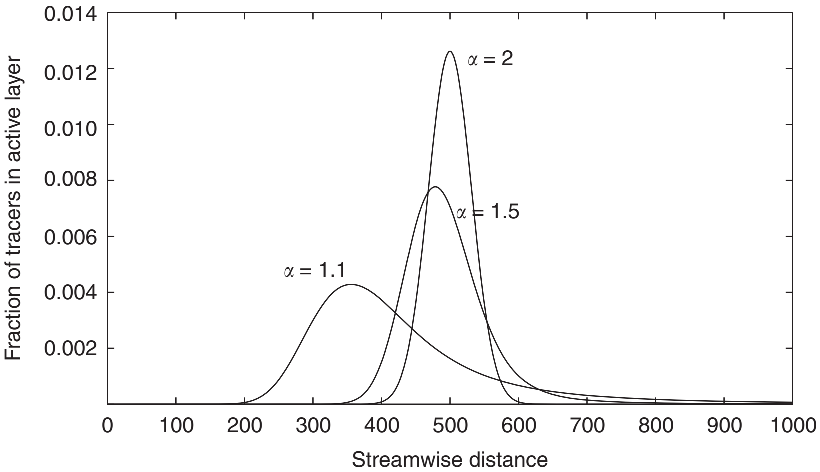


Figure 4 Long-time asymptotic solutions of the fractional advection–dispersion equation (ADE). Results are shown for three values of α (1.1, 1.5, and 2). Note that the parameter α controls how far the particles will disperse downstream of the source. After 500 days, only $\sim$5% of the tracers are recovered at 550-m downstream of the release point, according to the classical ADE model, whereas $\sim$8% and $\sim$18% are recovered according to the fractional ADE with $\alpha = 1.5$ and $\alpha = 1.1$, respectively. Reproduced from Ganti, V., Meerschaert, M.M., Foufoula-Georgiou, E., Viparelli, E., Parker, G., 2010. Normal and anomalous diffusion of gravel tracer particles in rivers. Journal of Geophysical Research 115, F00A12. doi:10.1029/2008JF001222, with permission from AGU.

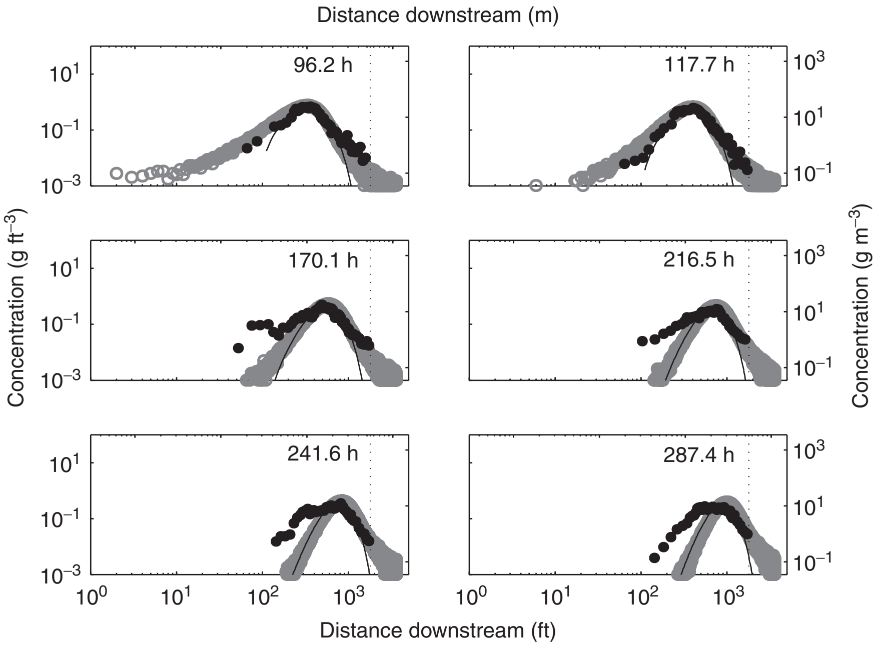


Figure 5 Comparison of the concentration profiles obtained with the fractional advection–dispersion model (gray circles) and the experimental observations of Sayre and Hubbell (black dots) and their classical advection–dispersion model (black line). Adapted from Bradley, D.N., Tucker, G.E., Benson, D.A., 2010. Fractional dispersion in a sand bed river. Journal of Geophysical Research 115, F00A09. doi:10.1029/2009JF001268, with permission from AGU.

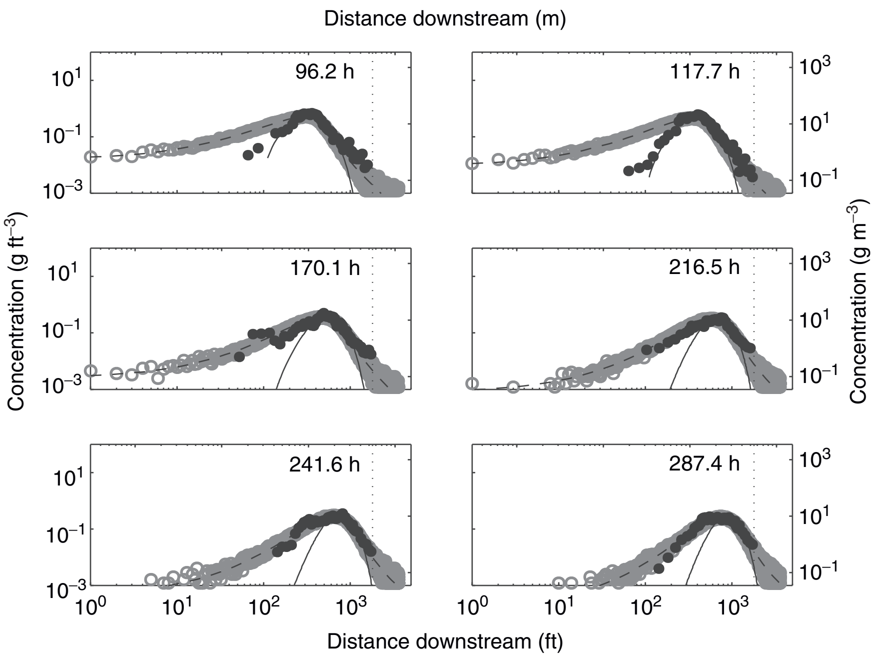


Figure 6 Comparison between the concentration profiles of the particle-tracking model (mobile phase plotted in gray) and the ones obtained with the fractional mobile/immobile formulation (dashed lines). Adapted from Bradley, D.N., Tucker, G.E., Benson, D.A., 2010. Fractional dispersion in a sand bed river. Journal of Geophysical Research 115, F00A09. doi:10.1029/2009JF001268, with permission from AGU.

2.8.4 Nonlocal Theories of Sediment Transport on Hillslopes

Starting from the qualitative observations of Davis (1892), several landscape evolution models have represented 1-D hillslope transport by linear diffusion:

$$q_s(x) = -K\nabla h(x) \qquad [18]$$

where $q_s(x)$ is the sediment flux [$L^3/L/T$] at location x, K is the diffusivity coefficient [L^2/T] and h is the elevation. By substituting eqn [18] into the continuity (Exner) equation:

$$\rho_r \frac{\partial h}{\partial t} = \rho_r U - \rho_s \nabla \cdot q_s \qquad [19]$$

where ρ_s and ρ_r are the bulk densities of sediment and rock and U is the rock uplift, we obtain the linear diffusion equation:

$$\frac{\partial h}{\partial t} = U + K\nabla^2 h \qquad [20]$$

where the bulk densities of sediment and rock have been assumed to be the same and chemical erosion has been ignored. According to this model, the flux is proportional to the local slope in a linear fashion (Gilbert, 1909; Culling, 1960; Hirano, 1968), resulting in equilibrium profiles of constant curvature:

$$\frac{d^2 h}{dx^2} = -\frac{U}{K} \qquad [21]$$

As in the case of tracer dispersal analyzed in the previous section, observed hillslope equilibrium profiles show deviation from those expected under the linear diffusion model. For example, soil-mantled hillslopes exhibit nonconstant curvatures and slopes are typically convex near the divide and increasingly planar downslope. Such variability is clearly not captured by the linear diffusion model.

Attempts to reproduce the observed hillslope profiles have led to the development of nonlinear formulations of hillslope transport, in which the sediment flux is proportional to the local slope in a nonlinear fashion (Kirkby, 1984, 1985; Anderson and Humphrey, 1989; Anderson, 1994; Howard, 1994a, 1994b, 1997; Roering et al., 1999). One such formulation is

$$q_s(x) = \frac{K\nabla h}{1 - (|\nabla h|/S_c)^2} \qquad [22]$$

where K is the diffusivity and S_c is a critical hillslope gradient (e.g., Roering et al. (1999)), both calibrated parameters. Figure 7 shows the comparison of the activity predicted by linear (thick line) and nonlinear (thin line) diffusion versus the experimental data (gray points) obtained from a hillslope in the Oregon Coast Range. As can be seen, the nonlinear transport law does a better job in reproducing the observed behavior than linear diffusion.

As we have seen, both the linear and nonlinear diffusion formulations are local in character, that is, the sediment flux is proportional to the local slope in a linear or nonlinear fashion. The nonlinear transport laws were introduced to be able to explain the deviation of equilibrium hillslope profiles from parabolic (expected from a pure diffusive transport or linear flux law) and also the observed nonlinearity of sediment flux to local gradients (see later discussion). However, what nonlinear models cannot take into account is the fact that heterogeneities in sediment-producing mechanisms upslope of the point of interest can result in a wide range of event-based downslope transport distances not captured by any local model.

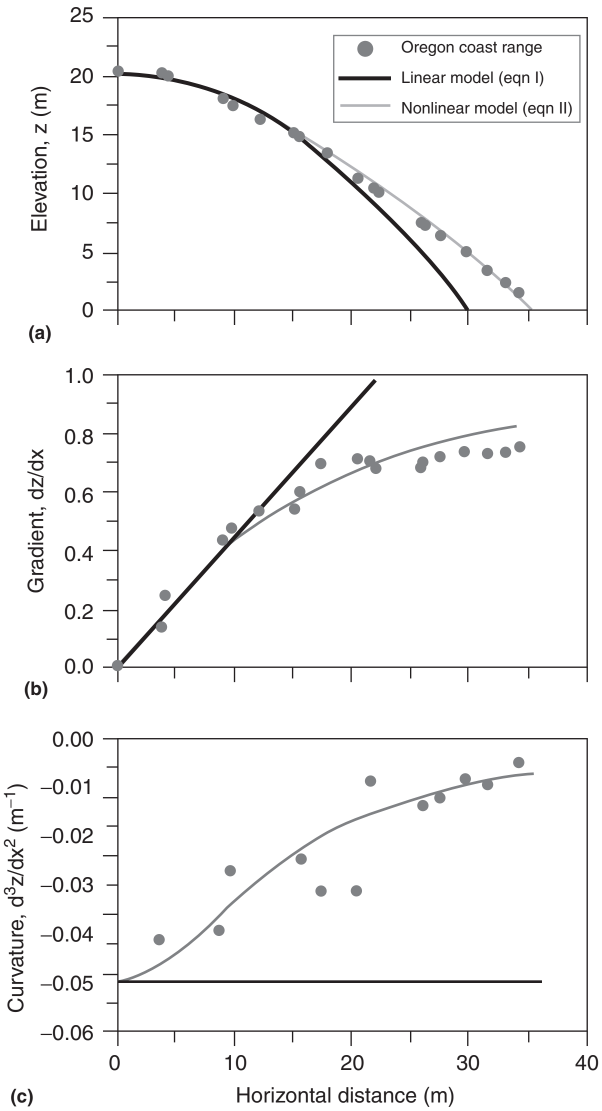


Figure 7 Comparison of the behavior predicted by linear (thick line) and nonlinear (thin line) plotted versus the experimental observations obtained in the Oregon Coast Range (gray dots) for (a) elevation, (b) gradient, and (c) curvature along a profile. The model predicted hillslopes are calculated assuming constant erosion. For the linear and nonlinear diffusion simulations, $K = 0.003\ \mathrm{m^2\,yr^{-1}}$, $\rho_r/\rho_s = 2.0$, and $S_c = 1.2$. Reproduced from Roering, J.J., Kirchner, J.W., Dietrich, W.E., 1999. Evidence for nonlinear, diffusive, sediment transport on hillslopes and implications for landscape morphology. Water Resources Research 35(3), 853–870, with permission from AGU.

Figure 8 illustrates some of the possible processes contributing to sediment transport on a hillslope, such as gopher mounds, tree throws, and wood blockage. Because a wide variety of length scales, corresponding to these processes, are involved, the number of particles arriving at a certain location downslope is affected by the upslope region.

To be able to take into account the heterogeneity of sediment producing processes and the corresponding length scales of transport, Foufoula-Georgiou et al. (2010) proposed a nonlocal flux-slope formulation. According to this

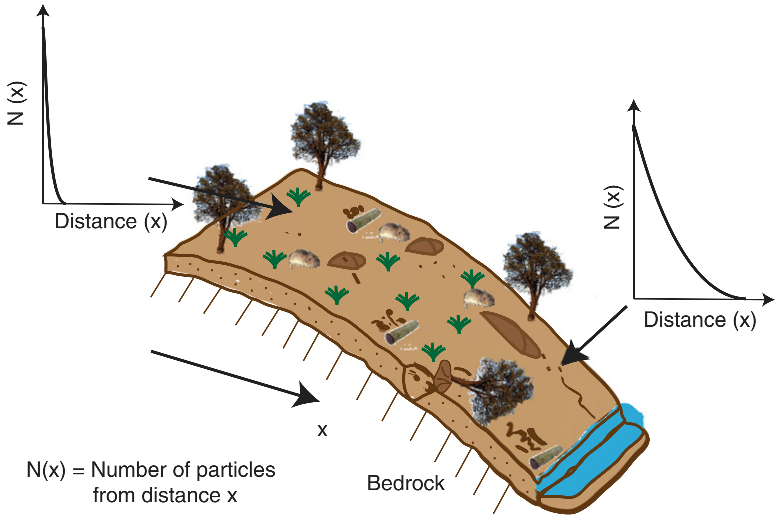


Figure 8 A wide variety of processes contribute to sediment transport on hillslopes: gopher mounds, tree throws, and wood blockage. Thus, the activity at a certain location downstream is affected by the processes acting in the upslope region. Reproduced from Foufoula-Georgiou, E., Stark, C., 2010. Introduction to special section on stochastic transport and emergent scaling on earth's surface: rethinking geomorphic transport – stochastic theories, broad scales of motion and non-locality. Journal of Geophysical Research – Earth Surface 115, F00A01. http://dx.doi.org/10.1029/2010JF001661, and Foufoula-Georgiou, E., Ganti, V., Dietrich, W.E., 2010. A nonlocal theory of sediment transport on hillslopes. Journal of Geophysical Research 115, F00A16. doi:10.1029/2009JF001280.

formulation, the sediment flux at a point is given by the weighted average of the upslope topographic attributes:

$$q_s^*(x) = -K^* \int_0^x g(l)\nabla h(x-l)\,\mathrm{d}l \qquad [23]$$

where $q_s^*(x)$ is sediment flux [L^3/L/T], K^* is the diffusivity coefficient, h is the elevation, and $g(l)$ is a kernel through which a weighted average of local gradients upslope of the point of interest is performed. The form of this kernel dictates, together with the continuity eqn [19], the final form of the hillslope profile evolution equation. In the case in which $g(l)$ decays as a power law with the lag l, then eqn [23] can be written in terms of a fractional derivative (Cushman and Ginn, 2000):

$$q_s^*(x) = -K^*\nabla^{\alpha-1}h(x) \qquad [24]$$

where the order of differentiation α varies between 1 and 2. Substituting the above expression for sediment flux into the continuity eqn [19], we obtain the governing equation for hillslope transport proposed by Foufoula-Georgiou et al. (2010), given by a fractional diffusion equation:

$$\frac{\partial h}{\partial t} = U + K^*\nabla^{\alpha}h \qquad [25]$$

Note that the order of differentiation α dictates the degree of nonlocality. For example, for the case $\alpha = 2$, we recover the case of linear diffusion and thus local dependence on slope. For $1 < \alpha < 2$, the transport is faster than linear diffusion and, as seen earlier, it is called 'superdiffusion'.

The hillslope profile at dynamic equilibrium is given by the solution to the following equation

$$\frac{\mathrm{d}^{\alpha}h}{\mathrm{d}x^{\alpha}} = -\frac{U}{K^*} \qquad [26]$$

By assigning the boundary conditions

$$h(0) = H_{\mathrm{top}} = \frac{U}{\Gamma(1+\alpha)K^*}L^{\alpha}$$

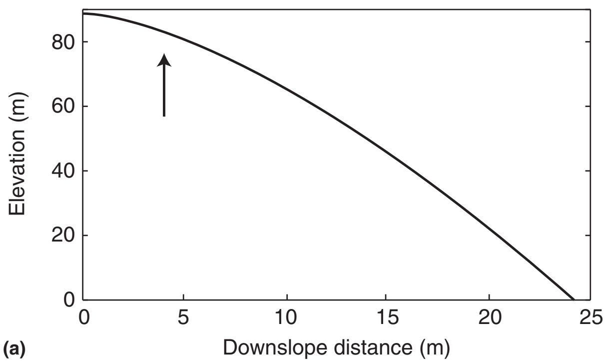
$$\left.\frac{\mathrm{d}h}{\mathrm{d}x}\right|_{x=0} = 0 \qquad [27]$$

Foufoula-Georgiou et al. (2010) solved eqn [26] numerically and obtained the steady-state equilibrium hillslope profile, which is shown in **Figure 9(a)** for the case of the degree of nonlocality $\alpha = 1.5$. Foufoula-Georgiou et al. (2010) obtained also the analytical solution of eqn [26] as

$$h(x) = H_{\mathrm{top}} - \frac{U}{\Gamma(1+\alpha)K^*}x^{\alpha} \qquad [28]$$

where h is the horizontal distance from the ridgetop, and H_{top} is the elevation of the ridgetop (the reader is referred to the original publication for subtle details on the boundary conditions for the numerical and analytical solutions). As can be seen from **Figure 9(a)**, the hillslope profile is parabolic near the ridgetop and becomes a power-law downslope with an exponent equal to the nonlocality parameter (consistent with the analytical solution; see **Figure 9(b)**). Note that the steady-state solution to the fractional governing equation predicts a power-law behavior for both the local gradient and curvature with downslope distance; thus, despite being a linear equation, it does not predict constant curvature as linear diffusion does, but a power-law decay with exponent dictated by the nonlocality parameter α.

To test the proposed fractional hillslope transport law, Foufoula-Georgiou et al. (2010) analyzed the behavior of three study sites. Note that, because of the 1-D character of the expression proposed, its applicability is limited to those cases in which transport can be assumed only along a specific



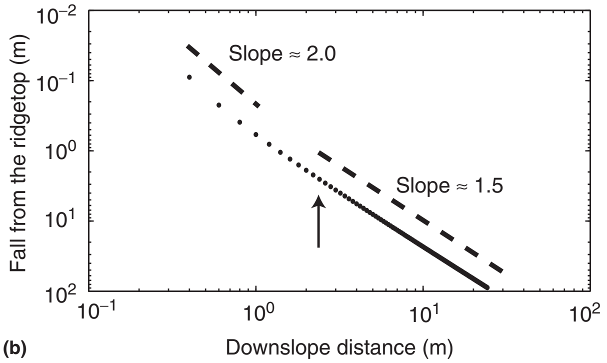


Figure 9 (a) Steady-state hillslope equilibrium profile obtained by the numerical solution of eqn [26] for the degree of nonlocality $\alpha = 1.5$, and (b) log–log plot of the vertical drop from the ridgetop versus downslope distance. The hillslope profile is parabolic near the ridgetop and later shows a power-law behavior with exponent α. Reproduced from Foufoula-Georgiou, E., Stark, C., 2010. Introduction to special section on stochastic transport and emergent scaling on earth's surface: rethinking geomorphic transport – stochastic theories, broad scales of motion and non-locality. Journal of Geophysical Research – Earth Surface 115, F00A01. http://dx.doi.org/10.1029/2010JF001661, and Foufoula-Georgiou, E., Ganti, V., Dietrich, W.E., 2010. A nonlocal theory of sediment transport on hillslopes. Journal of Geophysical Research 115, F00A16. doi:10.1029/2009JF001280.

profile, that is, profiles in which absence of significant planform curvature has been reported. The three study sites identified and analyzed are the Oregon Coast Range (see, e.g., Roering et al., 1999), the Shenandoah River area in Virginia (see Hack and Goodlett, 1960), and the grassland east of San Francisco, CA (McKean et al., 1993). Foufoula-Georgiou et al. (2010) report that all the three locations show noninteger exponents between 1 and 2 of the log–log plots of vertical drop from the ridgetop versus downslope distance, thus showing nonlocal behavior and the applicability of the proposed fractional transport law.

An important empirical observation that has motivated the development of local nonlinear hillslope transport models is the nonlinear dependence of sediment flux on local gradient. This dependence is clearly seen in **Figure 10**, which shows the variability of the sediment flux as a function of local gradient computed by Roering et al. (1999) for the Oregon Coast Range hillslope profiles within the small MR1 basin. An important additional observation from this figure is the natural variability of computed sediment flux for a given value of local gradient (i.e., vertical spread in **Figure 10**). In order to capture this variability, the calibration of the local nonlinear model of eqn [22] results in two sets of parameters as shown in **Figure 10**: K ranging from 0.0015 to 0.0045 $m^2\,yr^{-1}$ and the critical slope S_c ranging from 1.0 to 1.4. This parameter variability is significant within such a small basin, casting concerns about the physical interpretation of those parameters. An interesting question posed by Foufoula-Georgiou et al. (2010) was as to whether the linear nonlocal model of eqn [24] is able to reproduce the observed nonlinear dependence of sediment flux on the local gradient and whether it can do so with a more narrow range of parameter values. The sediment flux shown in **Figure 11** as a function of local gradient demonstrates that indeed this is the case, making the linear nonlocal model an attractive alternative to the typical nonlinear transport models. Besides, as demonstrated in Foufoula-Georgiou et al. (2010), by a simple Taylor series expansion, the nonlinear transport model of eqn [22] results in a linear diffusive term and a nonlinear quadratic term on the local gradient. As such, it essentially emulates superdiffusion by behaving as linear diffusion at low gradients, while accelerating diffusion (the nonlinear term dominates) in the presence of higher slopes. This behavior is concisely captured by the linear nonlocal transport model but for different physical reasons, that is, reflecting the upstream contribution to local sediment production due to natural heterogeneities, rather than a strict nonlinear dependence at the point of interest only, no matter what the upstream conditions are.

Having introduced the concept of nonlocality, the concept of an 'upstream influence length' arises naturally. In particular, one could ask how far upslope the computation of the local sediment flux has to go, such that all the relevant processes influencing the sediment flux at a certain location are included. Foufoula-Georgiou et al. (2010) defined the influence length L_α as the distance upslope from a certain location at which the contribution to the local sediment flux from beyond that distance drops to $<10\%$. Note that this value was arbitrarily chosen and will essentially depend on the characteristics of the landscape in analysis. The plot of L_α for several values of the degree of nonlocality α essentially shows what one would expect: the more nonlocal the nature of transport is (smaller values of α), the larger the value of L_α, meaning, the farther area upslope contributes to the computation of the local sediment flux.

The nonlocality in hillslope transport has also been recently tackled by Tucker and Bradley (2010), who proposed a discrete particle-based model instead of the continuum approach of Foufoula-Georgiou et al. (2010). Tucker and Bradley (2010) pointed out that the locality assumption is appropriate only when there is a clear gap between microscales associated with the motion of particles and macroscales associated with the system as a whole. In particular, they were interested in capturing the transition from local to nonlocal transport through the development of a particle-based hillslope evolution model. In their formulation, the hillslope is represented by a pile of two-dimensional particles that experience quasi-random motions. The scheme of the model is the following: after selecting at random a particle and a direction at each iteration, the particle is assigned a certain probability

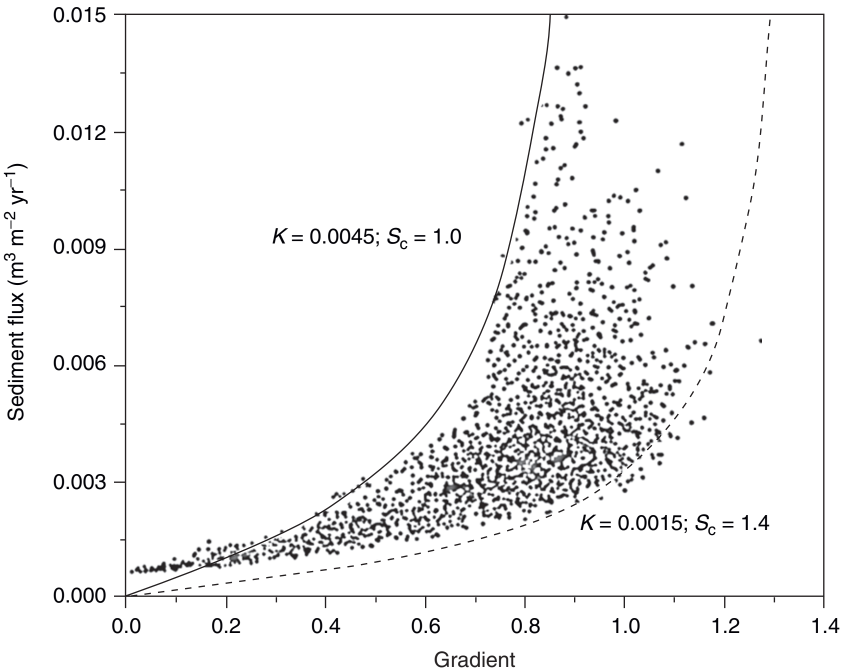


Figure 10 Variability of the sediment flux as a function of local gradient, computed on the Oregon Coast Range (MR1 basin hillslope). Note the nonlinear dependence of sediment flux on local gradient and the wide range of model parameters values that has to be employed for the nonlinear transport law to be able to capture the natural variability. Reproduced from Roering, J.J., Kirchner, J.W., Dietrich, W.E., 1999. Evidence for nonlinear, diffusive, sediment transport on hillslopes and implications for landscape morphology. Water Resources Research 35(3), 853–870, with permission from AGU.

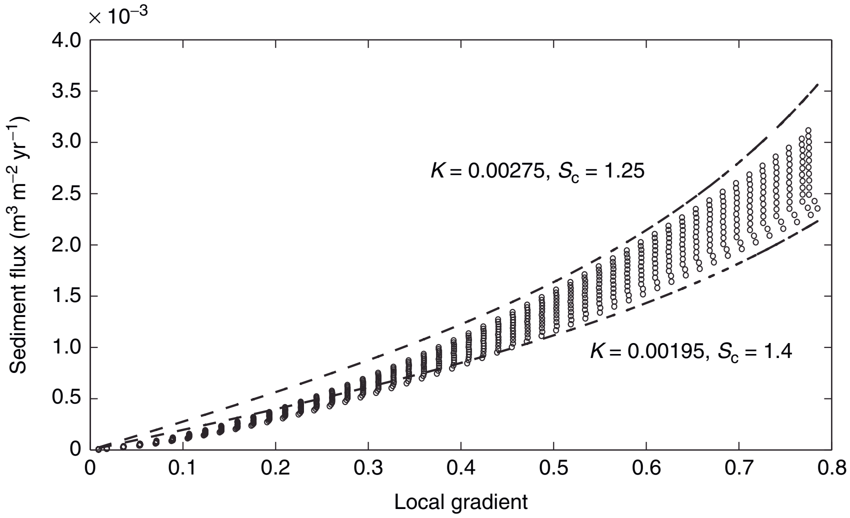


Figure 11 Sediment flux computed on several generated hillslope profiles, emulating the natural variability around the hillslope profile of Roering et al. (1999), and plotted as function of local gradient. Note that with a single set of parameters $K = 0.0007\ m^{\alpha}\ yr^{-1}$ and $\alpha = 1.5$, the variability observed in real hillslope profiles is captured. The broken lines show the fitted nonlinear model of eqn [22] needed to envelope this variability. Reproduced from Foufoula-Georgiou, E., Stark, C., 2010. Introduction to special section on stochastic transport and emergent scaling on earth's surface: rethinking geomorphic transport – stochastic theories, broad scales of motion and non-locality. Journal of Geophysical Research – Earth Surface 115, F00A01. http://dx.doi.org/10.1029/2010JF001661, and Foufoula-Georgiou, E., Ganti, V., Dietrich, W.E., 2010. A nonlocal theory of sediment transport on hillslopes. Journal of Geophysical Research 115, F00A16. doi:10.1029/2009JF001280, with permission from AGU.

of displacement in each direction, depending on local microtopography. Iterations are performed until the particle rests or exits the system. A schematic representation of the model is shown in Figure 12.

Tucker and Bradley (2010) employed several boundary conditions to be able to analyze the transition from local to nonlocal behavior in different settings: base-level lowering, scarp degradation, and cinder cone. In the case of base-level

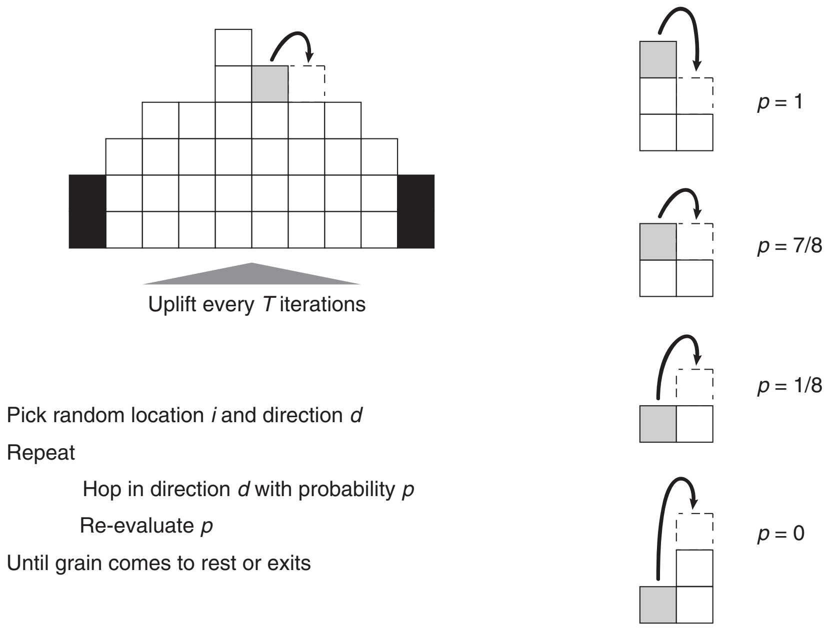


Figure 12 Schematic representation of the particle-based model proposed by Tucker and Bradley (2010). Reproduced from Tucker, G.E., Bradley, D.N., 2010. Trouble with diffusion: reassessing hillslope erosion laws with a particle-based model. Journal of Geophysical Research 115, F00A10. doi:10.1029/2009JF001264, with permission from AGU.

lowering, what Tucker and Bradley observed through the application of their particle-based model is that the behavior of the system at steady state (and thus whether of local or nonlocal character) essentially depends on the length of the domain of interaction of the particles and the velocity at which the system is evolved. In the case in which both length and velocity are small, the probability distribution function of displacement is thin tailed, namely an exponential distribution. This is translated into steady-state parabolic hillslope profiles and thus in the applicability of local linear diffusion. However, in the case in which the length and velocities are large, the probability distribution of displacement is broad and truncated at the system half length. This is translated into a nonlocal behavior, and, thus, the inapplicability of linear diffusion as the local gradient is less and less representative for the computation of the sediment flux at a point as the slope increases. Note that this dependence of local sediment flux on upslope length is a key ingredient of the fractional diffusion formulation proposed by Foufoula-Georgiou et al. (2010) and described earlier.

The second boundary condition of scarp degradation showed again a transition between an initially local behavior to a nonlocal behavior as the scarp widens, concluding with a diffusive-type behavior again as the gradient continues to shrink. Thus, Tucker and Bradley (2010) showed that a system can present both local and nonlocal behaviors, transitioning between the former and the latter as it evolves. The authors, thus, point out the need of developing geomorphic transport laws able to capture both local and nonlocal behavior at different spatial and temporal instances of the system evolution.

Another interesting point raised by Tucker and Bradley (2010) is the fact that, even within nonlocality, certain processes can be classified as strongly nonlocal, whereas others may be only weakly nonlocal. For example, even if fractional derivatives represent a powerful tool for modeling nonlocal transport, their applicability may be limited in cases in which particles move near the ground. In this case, as particle displacement is inevitably dependent on topography, the assumption of displacement statistics stationary in space and time, which is one of the basic assumptions behind the fractional approach, may fail. Through the concept of 'potential transport path', Tucker and Bradley (2010) also made a preliminary exploration of a continuum model that combines the particle-based approach and an analytical-continuum-type geomorphic transport law. The full development of this type of model is the domain of future research.

2.8.5 Nonlocal Landscape Evolution Models

Starting with the work of Culling (1960) regarding linear diffusion, several models have been proposed for landscape evolution with the idea of reproducing certain properties of natural landscapes. Following the distinction between modeling approaches proposed by Dietrich et al. (2003), models can be focused on making short-term predictions for specific features, for example, river grain size, or river bed depth (detailed realism); on large-scale predictions tested only on the appearance of the outcome, as knowledge of finer scale mechanisms is lacking due to computational restrictions (apparent realism); on reproducing statistical properties of the system emerging at steady state (statistical realism); and finally on explaining quantitative relationships, parametrizing models in terms of field measurements and observations (essential realism). All the approaches mentioned above have played an important role in understanding how landscapes are formed.

Most of the models require an assumption of a transport law and boundary conditions, and, then the solution of the conservation of mass over an initially random field until steady state is reached.

As mentioned earlier, when classical transport laws are implemented using local gradients and curvatures computed from digital elevation models (DEMs) of varying resolutions, the resulting sediment flux is expected to differ depending on the resolution (scale) of the DEM. This is because the larger the scale of the DEM, the 'smoother' is the topography perceived and thus the less the local magnitude and the system-wide variance of gradients and curvatures. Obviously, this scale dependence is not a desirable property of a model and poses the problem of parametrizing it in a way that does not require much calibration or tuning.

Passalacqua et al. (2006), by using a simplified model of erosion, showed the dependence of the model on the pixel resolution and developed a self-tuning subgrid-scale parameterization able to absorb the scale dependency. The landscape evolution model used involves a nonlinear dependence on the topographic gradient (Sornette and Zhang, 1993; Somfai and Sander, 1997; Banavar et al., 2001):

$$\frac{\partial h}{\partial t} = U - a \cdot q \cdot |\nabla h|^2 \qquad [29]$$

where h is the elevation, U is the uplift, a is the erosion coefficient, and q the water flux. Following the work developed in large eddy simulation of turbulent flows (e.g. Germano et al., 1991; Porté-Agel et al., 2000), Passalacqua et al. (2006) proposed a scale-dependent dynamic subgrid-scale parameterization, which can be shown to take the form of a power-law dependence on scale of the elevation gradients in the neighborhood of the point of interest. The subgrid-scale parameterization derived by Passalacqua et al. (2006) capitalizes on the assumption of self-similarity in topography elevation and 'borrows' the topography pixels surrounding a particular pixel to derive the local erosion coefficient. This is a way of introducing nonlocal information as one goes to a larger scale (larger vicinity around the point of interest) to incorporate information about the elevation gradient field around the point of interest.

We have seen that in all the cases presented in the previous sections, the nonlocal behavior was thought of arising from the heterogeneity of the contributing processes. In the context of landscape evolution modeling, the question is what type of heterogeneity mainly controls the landscape evolution behavior and how might a nonlocal behavior in this evolution arise. Passalacqua (2009) argued that this heterogeneity is brought about in this system by the wide variability of water flows contributing to erosion, as they are known to possess extreme fluctuations commonly manifested in heavy-tailed distributions of daily flows, maximum annual floods, or peak streamflow hydrographs (e.g. Gupta and Waymire, 1990; Dodov and Foufoula-Georgiou, 2004).

Passalacqua (2009) proposed to capture this variability by introducing a mathematical operation called 'subordination' (Bochner, 1949; Feller, 1971; Bertoin, 1996; Sato, 1999). This allows the mapping of the wide variability of landscape-shaping discharges to an equivalent wide variability of times over which landscapes could be eroded under the influence of a uniform unit discharge using the notion of

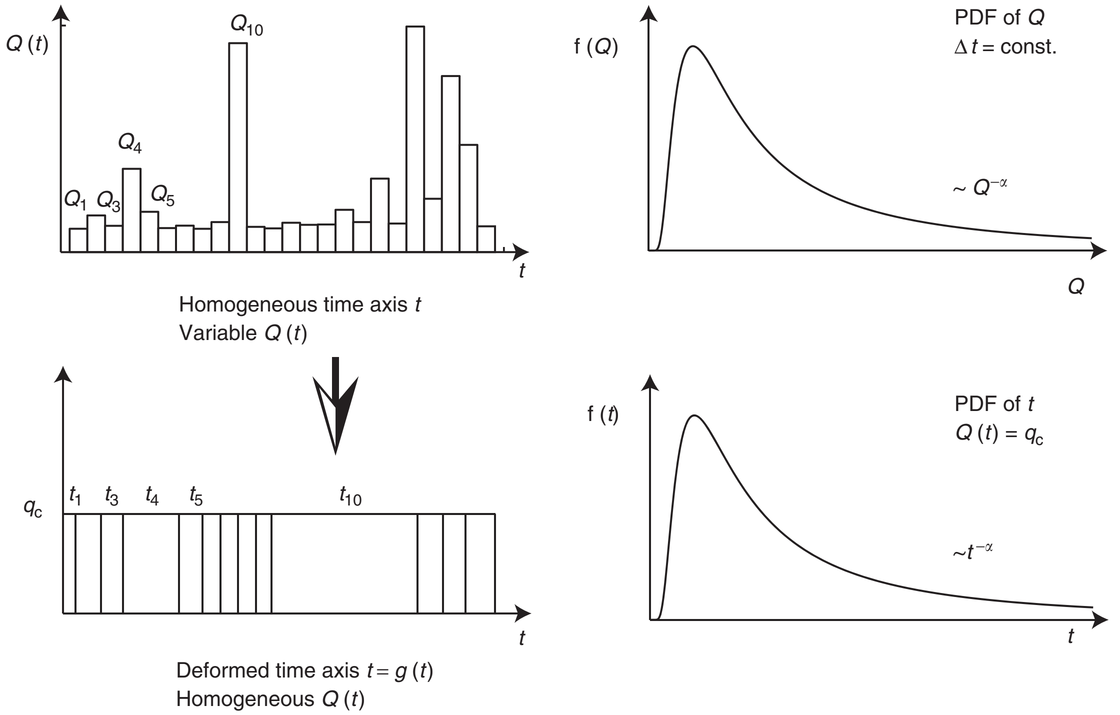


Figure 13 A highly variable streamflow applied over a homogeneous timescale is mapped into a uniform streamflow applied over a deformed timescale. The switch between 'real time' and 'operational time' is performed through the subordination operation.

'operational time'. This concept is illustrated schematically in **Figure 13**. The subordination operation consists in changing the 'real time' to 'operational time' and allows the mapping of a highly variable streamflow applied over a homogeneous time axis to a uniform streamflow applied over a deformed time axis.

Based on the earlier discussion, the linear nonlocal landscape evolution model proposed by Passalacqua (2009) is formulated as a fractional diffusion equation:

$$\frac{\partial h}{\partial t} = U - c \cdot q_c \frac{\partial^\alpha h}{\partial x^\alpha} \qquad [30]$$

where $1 < \alpha < 2$ is the order of differentiation. Note that the fractional diffusion process is a nonlocal operation along the flow paths, which explicitly acknowledges the fact that upstream geomorphic quantities have an effect on the erosion rate at any given location in the landscape.

After performing numerical simulations, Passalacqua (2009) compared the resulting steady-state landscapes of the nonlocal evolution model (eqn [30]) with $\alpha = 1.5$ to the ones obtained from the nonlinear model (eqn [29]) and to real topographic data from the Oregon Coast Range. The steady-state landscapes are compared in terms of appearance, slope–area relationships, and cumulative area distribution. As can be seen from **Figure 14**, the patterns arising from the nonlocal linear fractional diffusion model and the nonlinear model are very similar, indicating that nonlinearity is not a necessary ingredient for the formation of fluvial patterns, as usually stated (e.g. Birnir et al., 2001). Furthermore, Passalacqua (2009) reported that the log–log plot of the cumulative area distribution shows exponents in agreement with the ones found by other authors (Inaoka and Takayasu, 1993; Rinaldo et al., 1996; Sinclair and Ball, 1996; Banavar et al., 2001; Somfai and Sander, 1997) in all the three cases (results not shown here). Although the nonlinear model is theoretically bound to predict a 0.5 exponent of the slope–area relationship, the nonlocal formulation can accommodate a range of exponent values < 0.5, showing the potential of the fractional diffusion formulation in capturing the broader range of values of the slope–area relationship exponent observed in nature. It is noted that, although Passalacqua (2009) argued that the fractional diffusion formulation has the potential of absorbing the scale-dependent behavior observed in nonlinear transport laws, full demonstration of this assertion is the subject of current research, as more simulations are needed to be able to understand the effect of the nonlocality parameter α on the resulting steady-state landscapes.

2.8.6 Future Directions

In this chapter, we have presented a summary of recent developments in revisiting the classical geomorphic transport laws via the concept of nonlocality. This concept argues that the time and length scales of motion in natural geomorphic systems (from particle transport in a single stream, to sediment transport on hillslopes, and to landscape evolution) vary widely and cannot always be captured by flux computations, which depend on local properties of the system only in space and/or time. Rather, the upstream conditions and/or previous time state of the system are directly contributing to the local flux at the point of interest in space and time. One way of understanding this nonlocality is by realizing that the probability distributions of particle travel distances (or random transport velocities), as well as times of inactivity (e.g., times at which the particles are immobile) are generally heavy tailed, that is, they possess a power-law decay, as opposed to an exponential-type decay. In those cases, there is no clear separation between the scales of motion and the scale of the system itself and no local (linear or nonlinear) transport model can appropriately capture the system dynamics. Instead, formulations are needed that incorporate this heavy-tail stochastic variability in the system.

One such formulation is via the use of fractional derivatives, that is, via extending the classical ADE to a fractional ADE (fADE). For example, changing the second derivative in the diffusion equation to a fractional derivative of the order <2 yields a model of superdiffusion (particles spread faster than classical diffusion predicts), although changing the first-order time derivative to a fractional derivative of order <1 yields a subdiffusive behavior. The noninteger order of differentiation relates to the power-law tail parameter of the distributions of particle jump lengths and waiting times between jumps, respectively.

This review has concentrated on three specific problems: tracer transport in rivers, hillslope sediment transport, and landscape evolution modeling. It is pointed out that a recent collection of papers in the *Journal of Geophysical Research – ES* (see the introductory paper by Foufoula-Georgiou and Stark, 2010) includes other applications of nonlocal transport theories in geomorphology, including erosional–depositional systems (Voller and Paola, 2010), subsurface transport on hillslopes (Harman et al., 2010), landslide-driven erosion (Stark and Guzzetti, 2009), interpretation of sedimentary deposits (Schumer and Jerolmack, 2009), bedload transport (Hill et al., 2010), and bed deformation in sand-bed rivers (McElroy and Mohrig, 2009). Other applications of nonlocal transport theories include a coupled formulation of sediment buffering-bedrock channel evolution (Stark et al., 2009) and an extended nonlocal formalism of the Fokker–Planck equations (Furbish et al., 2009; see also Furbish and Haff, 2010).

The development of the new concept of nonlocality in approaching transport in geomorphic and near-surface hydrologic systems opens new avenues of research, but also dictates new observational requirements to be able to differentiate cause and effect in choosing nonlocal versus local theories for a particular system. As seen via the example of hillslope transport presented in this chapter, a similar nonlinear dependence of sediment flux to local gradient can arise equally well from a local nonlinear flux model or from a nonlocal linear model. Which one is to be chosen in a particular case? Also, it was seen that the presence of both long waiting times and large jump lengths cannot be resolved from the steady-state system behavior, as nonunique combinations of these two processes can result in the same single exponent for the system. This emphasizes the need to monitor in more detail the system behavior (not only its steady state) in order to adopt the proper equation, which characterized the system evolution.

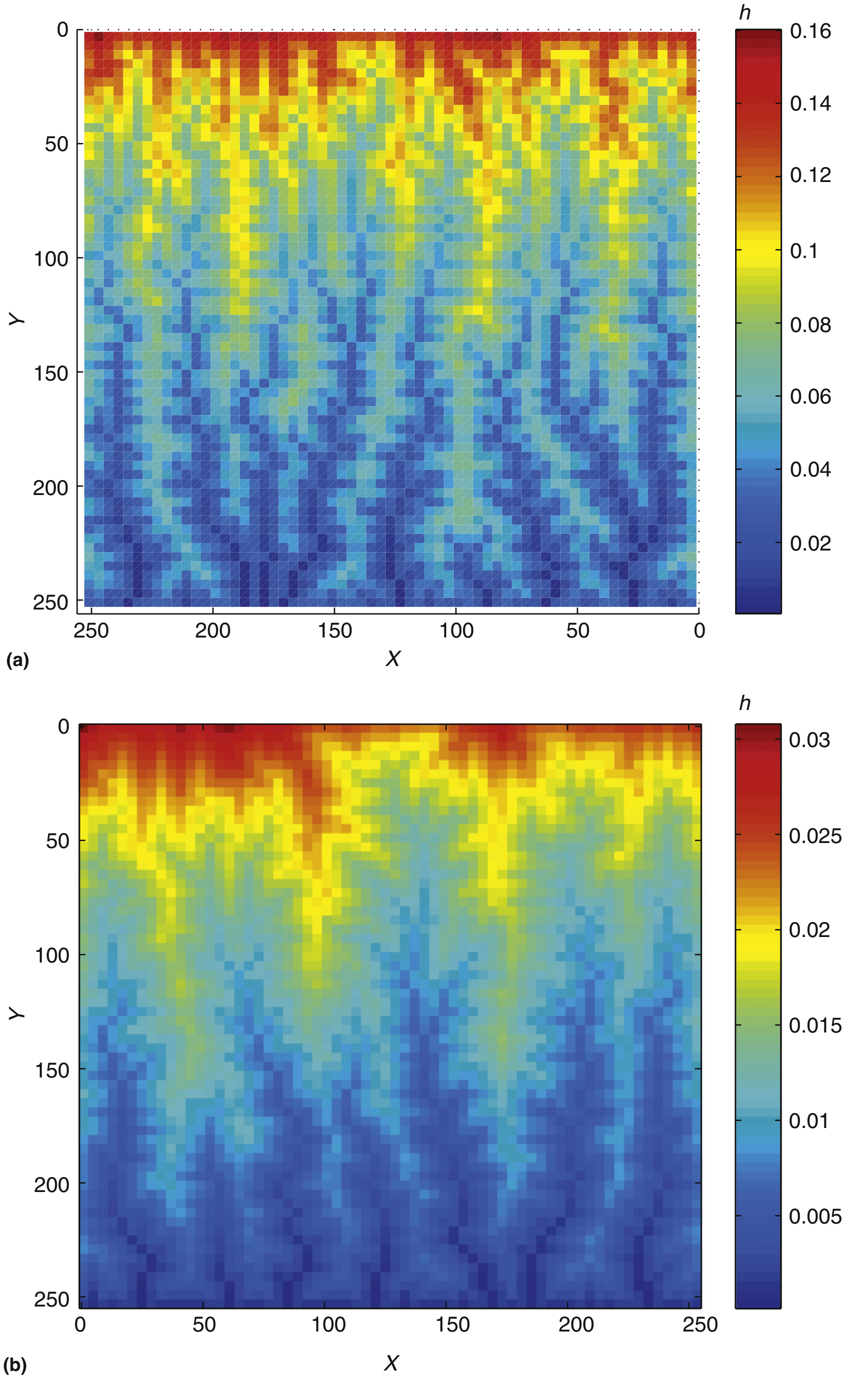


Figure 14 Steady-state fields obtained using the nonlinear formulation (a) and the nonlocal formulation (b). The patterns created are very similar, indicating that nonlinearity is not a necessary ingredient in the formation of fluvial patterns. Adapted with permission from Passalacqua, P., 2009. On the geometric and statistical signature of landscape forming processes. PhD dissertation, University of Minnesota, Minneapolis, MN, USA.

In this chapter, we alluded to some open problems that have emerged from the concept of nonlocal transport. First, nonlocal transport models have the potential to eliminate the scale-dependent sediment flux resulting from local linear or nonlinear models. More rigorous research is needed on this problem. Second, the notion of time subordination, that is, a time which is dynamically evolving, offers the potential to approach problems of transport in which large variability of flows over a long period of time can be equivalently modeled via a coarsened representation where the wide magnitude of flows is folded into an equivalent wide distribution of operational times of a constant flow. This is mathematically equivalent (Baeumer et al., 2009) to a fractional transport over flow paths, thus opening the door to nonlocal landscape evolution models, which may offer computational advantages, for example, avoid singularities which require very fine-grid computation. All these are issues for future research envisioned to occupy the geomorphologic research community over the next decade, especially as more and more high-resolution topographic data become available (Passalacqua et al., 2010), and new observational capabilities allow the accurate dating of surfaces (e.g., radionuclides), thus providing a closer monitoring of surface evolution.

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