Geomorphic Transport laws: Non-local Flux with Classical Mass Balance

By

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ABSTRACT

Most geomorphic transport laws proposed to date are local in character, i.e., they express the sediment flux at a point as a function of the elevation gradient or curvature or other geomorphic quantities at that point only. We argue that non-local constitutive laws, in which the flux at a point depends on the conditions in some larger neighborhood around this point in space and/or time, present a physically-motivated alternative which can handle the presence of heterogeneities known to exist in landscapes over a large range of scales. A particularly attractive subclass of these non-local constitute laws involves fractional (non-integer) derivatives in time and/or space and provides a rich class of models extensively studied in other fields of science. We draw on two examples to illustrate the scale dependency of the parameters of local geomorphic transport equations and suggest that this dependency has the potential to be alleviated by non-local constitutive laws.

INTRODUCTION

Paraphrasing Dietrich et al (2003), we use the term “geomorphic transport law” to refer to a mathematical statement that expresses the flux or erosion caused by one or more processes, in a manner that it can be parameterized from field observations, it can be verified, and it can be applied over geomorphically significant spatial and temporal scales. Although geomorphic transport laws have their origin in physical laws and mechanisms, they are not always fully derivable from first principles. Rather, they are minimum complexity models that are physically-motivated and capture the essential features of landscapes.

A starting point of geomorphic transport theories is the conservation of mass which (assuming a constant bulk density for simplicity) says that the divergence of the sediment transport vector is balanced by the storage and production of sediment:

$$-\nabla \cdot q_s = \frac{\partial h}{\partial t} - P \tag{1}$$

or

$$\frac{\partial h}{\partial t} = P - \nabla \cdot q_s \tag{2}$$

where h is elevation of the ground surface (or soil or sediment thickness), P is a sediment production term, and q_s is the sediment flux.

The simplest, and still most commonly used, sediment flux expression was proposed by Culling (1960) in analogy to Fick’s law of diffusion. The conventional Fickian constitutive theory maintains that the dispersive flux is proportional to the gradient of the concentration field and thus, by analogy, the sediment flux was considered to be proportional to the topographic gradient:

$$q_s = D \cdot \nabla h \tag{3}$$

where D is the diffusivity coefficient. Coupling this equation with the conservation equation results in the standard diffusion equation:

$$\frac{\partial h}{\partial t} = P - \nabla \cdot (D \cdot \nabla h) = P - D \cdot \nabla^2 h \tag{4}$$

It is easy to see that the linear transport law of (3) and the diffusion equation it results in, imply that under equilibrium conditions $(\partial h / \partial t = 0)$ the hillslope profiles will have a constant curvature at all points along the profile, and thus a parabolic shape. Field observations do not support this purely diffusive behavior in many hillslopes and have prompted the proposal of more complex transport laws which have a nonlinear dependence of sediment flux on topographic gradient. A review of several of these laws can be found in Dietrich et al. (2003). For example, for soil mantled hillslopes, Roering et al (1999) proposed transport that varies linearly with slope at low gradients but increases nonlinearly as slope approaches a critical value. In particular, using the balance between frictional and gravitational forces in a soil undergoing disturbance they derived the following transport equation:

$$q_s = \frac{K \cdot \nabla h}{1 - (|\nabla h| / S_c)^2} \tag{5}$$

where K is a transport coefficient (involving the power expenditure per unit area, soil bulk density, coefficient of friction, and gravitational acceleration), and S_c is a critical hillslope gradient (equal to the effective coefficient of friction) both of which have to be estimated from field observations. Similar equations have been derived before by others (e.g., Andrews and Buckman, 1987; Howard 1994; see also the review of Dietrich et al., 2003). It is interesting to note that the above mechanistically-derived expression for sediment flux is identical to the so-called Perona-Malik non-linear diffusivity model:

$$\frac{\partial h}{\partial t} = \nabla \left(\frac{\nabla h}{1 + \frac{|\nabla h|^2}{k^2}} \right) \tag{6}$$

where $k > 0$ is a parameter, used to preserve both fine scale features (edges) and large scale features in image processing of DEMs (e.g. see Braunmandl et al., 2003).

For other than hillslope fluvial transport systems, e.g., bedrock or alluvial channel incision, landslide transport, Horton overland flow erosion, river sediment transport etc., similar expressions for sediment transport have been proposed. Several of these transport laws end up expressing q_s as a function of local gradient, upstream area (a surrogate for flow), shear stress and other geomorphic parameters (see Tucker and Bras, 1998 and also Dietrich et al., 2003 for a review).

We note that the geomorphic transport laws existing to date are all local constitutive laws, i.e., they relate the sediment flux at a point in space and time to the elevation gradient or other quantities at that same point. We argue in this paper that this locality is a limiting factor both theoretically and practically. Theoretically, because the extension of the classical definition of divergence (which applies as the control volume shrinks to zero) to a finite-size volume is ill-defined for a medium that exhibits heterogeneities at all scales (e.g., see Benson, 1998). Practically, because the manifestation of the above inappropriate extension is a dependence of the model parameters on scale which presents a problem in practical applications using DEM data.

We propose a deviation from local geomorphic transport theories, namely the exploration of non-local constitutive laws which consider that the flux at a point depends on the conditions in some larger neighborhood around this point in space and/or time. For example, the flux at a point can be considered to be a weighted average of the topographic gradients in this larger space-time neighborhood. It is shown that this nonlocal flux notion is equivalent to introducing a non-integer order (fractional) notion of divergence of sediment flux. This allows the exploration of a large class of fractional (pseudo-differential) models that have been explored extensively in the literature for modeling subsurface transport, hydrodynamics, statistical mechanics, molecular biology, and turbulence (e.g., see Bouchaud and Georges, 1990; Pekalski and Sznajd-Weron, 1999; Shlesinger et al, 1995 and references therein).

DIVERGENCE OF THE SEDIMENT FLUX VECTOR

The advection-dispersion equation (ADE) is based on the classical definition of divergence of a vector field. The divergence is defined as the ratio of total flux through a closed surface to the volume enclosed by the surface when the volume shrinks to zero (e.g., Schey, 1992; see also Benson, 1998, for a nice exposition relevant to subsurface transport)

$$\nabla \cdot q_s = \lim_{V \to 0} \frac{1}{V} \iint_S q_s \cdot \eta dS \tag{7}$$

where q_s is a vector field, V is an arbitrary volume enclosed by surface S, and η is a unit normal vector. Implicit in this equation is that the limit of the integral exists, i.e, the vector q_s exists and is smooth as $V \to 0$.

The classical notion of divergence maintains that as an arbitrary control volume shrinks, the ratio of total surface flux to volume must converge to a single value. However, the sediment flux is primarily due to fluctuations in topography (i.e., elevation gradients) which are known to exhibit variability down to very small scales including abrupt changes and discontinuities. For this reason, the classical divergence theorem is of little use in geomorphology. Rather, a divergence associated with a finite volume and defined as the first derivative of total flux to volume is more relevant. However, one notes that by increasing the arbitrary volume, a larger variability of topographic gradients is sampled and it is known that this variability depends on scale. Thus, the ratio of total flux to volume does not remain constant but varies with the size of the volume. As a result, the classical diffusion equation is no longer self-contained with a close form solution at all scales. To adopt the classical theory, the best approximation that can be done is to assume that the total flux to volume can be assumed piece-wise constant within small ranges of scales, allowing one to talk about an "effective" scale-dependent dispersion coefficient.

Several techniques have been proposed in the subsurface transport literature to tackle the problem of scale-dependent dispersivity which arises for similar reasons, namely, the presence of inhomogeneities at all scales, or the fractality of the porous medium. These vary from small perturbation approaches and effective parameterizations (e.g., Gelhar and Axness, 1983; Dagan, 1997), to power law dependence of D on scale (e.g., Su, 1995), to volume statistical averaging (e.g., Cushman, 1991, 1997) and to fractional advection dispersion equations (FADE) (e.g., Benson, 1998; Benson et al., 2001; Bauemer et al., 2001; and Schumer et al., 2001). A review of these methodologies can be found in Benson (1998).

NONLOCAL GEOMORPHIC TRANSPORT LAWS

We propose the introduction of a non-local conservation law which (omitting the sediment production term of equ 1 for simplicity) reads:

$$\frac{\partial h}{\partial t} = -\nabla \cdot q_s^* \tag{8}$$

where the sediment flux q_s^* at a point (x,t) does not depend on the elevation gradient at that point only but on the elevation gradients at a collection of surrounding points, i.e.,

$$q_s^*(x,t) = \int_0^t \int_{-\infty}^{\infty} D(l,\tau) \nabla h\left(x-l, t-\tau\right) dl d\tau \tag{9}$$

In the above expression, the dispersion coefficient is a function of spatial and temporal lags and the flux can be seen as a convolution of a "generalized dispersion coefficient" with the elevation gradient field. In other words, this generalized dispersion coefficient (called the "memory kernel") introduces a memory effect that accounts for the contribution of the surrounding points to the sediment flux at the local point (x,t). This type of dispersion has been termed "convolution-Fickian" dispersion by Cushman (1991; 1997).

There are several forms of the memory kernel that can be explored but one form of particular interest is a power law function on the lag. Considering only spatial memory (extension to temporal memory is also possible), this kernel reads:

$$D(l,\tau) \sim l^{2-\alpha} \tag{10}$$

It is noted that for $\alpha = 2$ there is no memory (reducing to the local conservation law), and for values $1 < \alpha \le 2$ there is a power law decrease of the influence of surrounding points. As is shown below (following Cushman and Ginn, 2000) this specific form of the kernel is equivalent to considering a fractional (non-integer) notion of divergence of flux of the form

$$q_s^* = D \cdot \nabla^{\alpha-1} h \tag{11}$$

and thus a non-Fickian fractional dispersion equation of the form

$$\frac{\partial h}{\partial t} = \nabla \cdot (D \cdot \nabla^{\alpha-1} h) = D \cdot \nabla^{\alpha} h \tag{12}$$

To be able to prove the above assertion, we need to define fractional derivatives. The fractional derivative $d^{\alpha}/(dx^{\alpha})$ of a function $f(x)$ in one dimension and for $1 < \alpha \le 2$ is defined as:

$$\frac{d^{\alpha} f(x)}{dx^{\alpha}} = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^x \frac{f(x-y)}{y^{\alpha+1-n}} dy \tag{13}$$

with n the smallest integer larger than α (in our case then, $n = 2$). After some algebra (see Cushman and Ginn, 2000) one can show that

$$\frac{d}{dx}\int_0^t \int_{-\infty}^{+\infty} D(l,\tau)\frac{d}{d(x-y)}h(x-l,t-\tau)dld\tau = D\frac{d^\alpha h}{dx^\alpha} \tag{14}$$

when the kernel $D(l,\tau)$ takes the form:

$$D(l,\tau) = D\frac{\delta(\tau)H(l)}{\Gamma(2-\alpha)}l^{1-\alpha} \tag{15}$$

with $\delta(\tau)$, $0 \le \tau \le t$, the Dirac delta, $H(l)$ the Heaviside function on $(0,\infty)$, and $\Gamma(.)$ the Gamma function.

There is an interesting observation to be made. Namely, that the non-local conservation law (eqs 8 and 9) with memory on the surrounding points such that each local elevation gradient is appropriately weighted with a dispersion coefficient $D(l,\tau)$, has now been replaced with a "local" conservation equation (eq. 12) with a single constant dispersion coefficient D but with a non-integer differential operator on elevation $\nabla^a h$.

The above integral form of non-local constitutive laws with power law memory kernel, is of course a special case of the general theory. However, it results into a very important class of models (fractional advection dispersion equations – FADE and also further extensions that include nonlinear terms, as will be discussed later) that has been extensively studied and a wealth of theoretical developments and practical applications exist to draw upon. Besides, it is a natural class for geomorphology for two reasons. First, if one accepts that elevation fields exhibit variability down to very small scales and specifically that they have a variance which exhibits power law scaling (e.g., see example spectrum in Passalacqua et al., 2006 and also in Rodriguez-Iturbe and Rinaldo, 1997), then mathematically the first derivative of this (fractal) field does not exist. Rather one has to "regularize" this field by an integrating or smoothing kernel to be able to take derivatives (notion of non-local flux computation above) or, equivalently, to take derivatives not of degree one but of a lesser (fractional) degree, as much as allowed by the fractality of the elevation field. Second, as will be demonstrated below, the use of classical (local) conservations laws in geomorphology results in a power law dependence of the coefficients in order for the flux to be preserved, which as seen theoretically above suggests the use of fractional transport equations for this scale-dependency to be eliminated.

SCALE-DEPENDENCY OF CLASSICAL GEOMOPRHIC TRANSPORT LAWS

When the classical transport laws are implemented in practice using "local" gradients and curvatures computed from Digital Elevation Models (DEMs) of varying resolutions, the resulting sediment flux is expected to differ depending on the resolution (scale) of the

DEMs. This is because, the larger the scale of the DEMs, the "smoother" the topography is perceived and thus the less the variance of the gradients and curvatures (see for example, Lashermes et al., 2007 for the way the pdfs of curvatures change as a function of the scale). Obviously, this is not a desirable effect (see also Stark and Stark, 2001) and poses the problem of understanding this scale-dependency and parameterizing it in a way that does not require much calibration or tuning.

This problem was recently addressed in Passalacqua et al. (2006). Using a simplified model of erosion, corresponding to the so-called Burgers equation (e.g., see Somfai and Sander, 1997; Sornette and Zhang, 1993; Banavar et al., 2001; Pelletier, 2004, 2007 for use of this model in landscape evolution) a dynamic self-tuning methodology of deriving scale-dependent parameters using principles of dynamic sub-grid scale parameterization was proposed. The landscape evolution model used involves a nonlinear dependence on topography gradient:

$$\frac{\partial h}{\partial t} = U - a \cdot q \cdot |\nabla h|^2 \tag{16}$$

where q is the water flux, a is a parameter and U is a constant uplift rate. As shown in Passalacqua et al. (2006) keeping the parameter a constant and changing the resolution of the DEM data resulted in sediment flux from the domain boundaries which was strongly dependent of scale (see Figure 6 of that paper). Akin to the dynamic sub-grid scale parameterization formulations used in turbulence (e.g. Gernamo et al., 1991) , it was proposed that the parameter a is not kept constant but rather made scale-dependent in the following way:

$$\alpha_{\Delta} / \alpha_{\Delta/2} = \alpha_{2\Delta} / \alpha_{\Delta} = \beta \tag{17}$$

where now β is a constant. By applying equation (16) at the coarse-grained (filtered) elevation field at scales Δ and 2Δ, the coefficient β was derived to be:

$$\beta = \frac{\left\langle \overline{\left|\nabla \tilde{h}_{\Delta}\right|^2} \right\rangle}{\left\langle \left|\nabla \tilde{h}_{2\Delta}\right|^2 \right\rangle} \tag{18}$$

where $<\bullet>$ denotes spatial averaging over the entire field, $\widetilde{h_{\Delta}}$ denotes the filtered field at scale Δ and the overbar denotes spatial averaging using a filter of 2Δ (see Passalacqua et al., 2006 for derivation).

It is important to note that in essence the above expression informs the local coefficient at scale Δ by the elevation gradients at scales Δ and 2Δ and by a proportionality factor that accounts for the ratio of the statistics of elevation gradients over the whole field at these two scales. In a sense, one can see this as an attempt to introduce non-local information as one went to a larger scale (larger vicinity around the point of interest) to borrow information about the elevation gradient field. Using this new coefficient, the sediment flux dependence on scale was alleviated somewhat but not completely (see Fig. 7 of that paper).

A further refinement was proposed by following the dynamic sub-grid scale approach of Porte-Agel et al. (2000) in which the assumption was made that β is not constant anymore but that the ratio $\beta_\Delta / \beta_{2\Delta}$ is constant. This latest assumption implies that one goes out to even larger scales to get information about the local value of the coefficient a_Δ. Specifically, as is shown in Passalacqua et al., (2006), the coefficient α_Δ takes the form

$$\alpha_\Delta = \frac{\beta_{2\Delta}^2}{\beta_{4\Delta}} \cdot \alpha_{\Delta/2} \tag{19}$$

i.e., one involves scales 2Δ and 4Δ into deriving the coefficient at scale Δ. The exact expressions for deriving adaptively those coefficients as the landscape simulation proceeds can be found in Passalacqua et al. (2006). This new approach completely eliminated the scale dependence of the sediment flux as can be seen in Fig 8 of that paper.

The point we make here is that the sub-grid scale parameterization approach capitalizes on the assumption of a self-similarity in topography elevations and “borrows” the topography pixels surrounding a particular pixel to derive the “local” coefficient of the transport law. Moreover, it does this adaptively as the model evolves over time and space. This approach is fine when a landscape evolution model is to be used for simulation purposes but it is not helpful when one has to compute sediment transport from a particular landscape. Besides, it is important to point out that in both of the above sub-grid scale parameterization approaches, the form of the coefficient of the nonlinear term can be shown to be expressable as a power law on scale Δ, where the exponent and pre-exponent are computed adoptively. For example, equ (17) implies that $\alpha_\Delta \sim \Delta^{\log_2 \beta}$ with β as given by equ (18). We suggest that this provides another hint that a power law dependent memory kernel in the definition of non-local flux (equ 15), and thus the use of fractional derivatives in sediment flux, might be a natural choice or at least not an inconsistent choice with the scale dependent coefficients derived above to account for the sub-grid scale variability.

NONLINEARITY OR FRACTIONAL DIFFUSION OR BOTH?

In this section we make an interesting observation. Namely that the nonlinear transport model (equ 5) derived by Roering et al. (1999), to explain hillslope forms that do not conform to standard diffusion, ends up having a coefficient of the nonlinear term which is scale dependent and specifically has a power law dependence on scale. Performing a Taylor series expansion on the equation of Roering et al. (equ 5), one obtains

$$q_s \simeq D \cdot \nabla h + (D / S_c^2) \nabla h \cdot |\nabla h|^2 + \ldots \tag{20}$$

and inserting this form into the continuity equation, one obtains

$$\frac{\partial h}{\partial t} \simeq D \cdot \nabla^2 h + \lambda |\nabla h|^2 \tag{21}$$

with

$$\lambda = (D \big/ S_c^2) \nabla^2 h \tag{22}$$

It is observed that the coefficient of the nonlinear term is proportional to the local curvature. First, local curvature is a second order derivative of the elevation field and as such the above formulation suggests that in computing it from DEMs one has to go further than the local point (to at least twice the resolution of the grid size) to estimate the coefficient of the nolinear term. Second, it is well known that computation of "local curvature" from very high resolution DEMs (e.g., 1m LIDAR data) requires a smoothing in the vicinity of the point of interest to obtain robust estimates. For example, Roering et al. (1999) used a second degree polynomial fitted around each pixel over areas of the order of 10 m, and then computed the curvature from the coefficients of this polynomial. In a recent study of Lashermes et al. (2007) a wavelet-based formalism was proposed for computation of local curvature at different scales. As was explained in that study, this "local" curvature was really a non-local quantity which was computed as a weighted average of elevations at surrounding pixels with weights proportional to a kernel (in that study this kernel was the second derivative of the Gaussian function, the so-called Mexican hat wavelet). It was also shown that depending on the width of the kernel (neighborhood around the point of interest) the statistics of the curvature changed and in fact, the variance of curvature ended up having a power law dependence on the kernel width (see Fig. 1 of that paper).

Putting it all together, we suggest that the semi-mechanistically derived transport law of Roering et al., which by the way tried to balance the upslope and downslope transport components so inherently included a notion of non-locality, ends up having a coefficient of the nonlinear term which depends on scale and specifically in a power law way. This is similar to the adjustment of the coefficient of the nonlinear term in Passalacqua et al., (2006) and both point to the observation that the heterogeneity and particularly the self-similarity (fractality) of landscapes supports the adoption of a non-local conservation law with a memory kernel of power law form (equ 12) which gives rise to a fractional (non-integer) transport equation.

It is noted that although the theoretical results presented in section 2 were for dispersion only, both geomorphology examples involved a non-linear term. It goes beyond the scope of this paper to discuss the more general fractional non-local transport laws which include both a fractional power in the Laplacian Δ (fractional dispersive term) and a general algebraic nonlinearity (e.g., Biler et al., 1998). These laws take the form

$$\frac{\partial h}{\partial t} = -D(-\Delta)^{\alpha/2} h - \lambda(\nabla h)^r \quad (23)$$

with $\alpha \in (0,2], r \geq 1$ and λ a fixed parameter and are extensively discussed in Woyczynski (1998), and also Biler et al. (1998, 2001). It is noted that the above equation reduces to the standard advection-dispersion equation for $\alpha = 2$ and $r = 1$ and to the classical Burgers equation for $\alpha = 2$ and $r = 2$. Of interest in most applications is the so-called the "critical" case, when the diffusion and the nonlinear terms balance, that is, are of the same importance over the entire time scale. In this case, it can be shown (see Biller et al., 1991) that the parameters satisfy the equation $r = 1 + (\alpha - 1)/n$,with n is the dimension of the flux vector, and that the equation admits a self-similar solution. These models are of specific interest in geomorphology due to their scale invariance properties as will become apparent in the next section.

A STOCHASTIC POINT OF VIEW

It is noted that the geomorphic transport laws discussed in section 1 are deterministic and physically or mechanistically motivated. There is another class of models that has been used for landscape evolution motivated by the literature on stochastic growth equations. In physics, chemistry etc. a useful approach in understanding the behavior of various growth processes has been to derive the underlying continuum equation (stochastic differential equation) for the process under study following some symmetry pronciples (e.g., see Hwa and Kardar, 1992 and Barabasi and Stanley, 1995). The guiding principle is that the derived continuum equation is the simplest possible equation compatible with the symmetries of the problem. In general, the growth of an interface can be described by the continuum equation:

$$\frac{\partial h(x,t)}{\partial t} = G(h,x,t) + \eta(x,t) \quad (24)$$

where $G(h,x,t)$ is a general function that depends on the interface height, position and time and $\eta(x,t)$ is a noise term describing the roughening of the surface by a random process. It is noted that higher order derivatives in time are ignored since they can be shown to be irrelevant to the long-term behavior of the system (see Barabasi and Stanley, 1995). Using basic symmetry principles, i.e., invariance under time translation, translation invariance along and perpendicular to the growth direction, rotation and inversion symmetry about the growth direction and up-down symmetry for h) the simplest equation describing the equilibrium interface is the so-called Edwards-Wilkerson (EW) equation

$$\frac{\partial h(x,t)}{\partial t} = \nu \cdot \nabla^2 h + \eta(x,t) \quad (25)$$

The parameter ν (akin to dispersivity) is often called "surface tension" since the term $\nu \cdot \nabla^2 h$ tends to smooth the surface by redistributing its irregularities on the interface while maintaining average height. Note that a non-zero velocity of the interface (u) can be added to the above equation in the RHS term. The EW equation is a linear equation (diffusion with external noise).

The first extension to include nonlinear terms was proposed by Kardar Parisi and Zhang (KPZ; see Kardar et al., 1986). The construction of the KPZ equation follows the symmetry principles discussed above except that the up-down symmetry of the interface $h(x,t)$ is broken. The source of this symmetry breaking is a force (not necessarily external) perpendicular to the interface, which selects a particular growth direction for the interface. The physical motivation is to allow lateral growth, i.e., growth in the direction of the local normal to the interface. It can be shown that to include the presence of lateral growth, terms such as $(\nabla h)^{2n}$ must be added to the growth equation (see Barabasi and Stanley, 1995). The lowest order term of this sort is the nonlinear term $(\nabla h)^2$ which added to the EW equation results in the KPZ equation

$$\frac{\partial h(x,t)}{\partial t} = \nu \cdot \nabla^2 h + \frac{\lambda}{2}(\nabla h)^2 + \eta(x,t) \tag{26}$$

where the first term in the RHS describes the surface smoothing, the second nonlinear term is caused by the slope-dependence of the surface growth rate and the third term is a noise term. It is noted that the presence of a nonlinear term makes the mean velocity of the interface

$$u = \frac{\lambda}{2}\int_0^L d^2x < (\nabla h)^2 > \tag{27}$$

nonzero even in the absence of an external driving force. The geometrical interpretation of the nonlinear term can be understood by noting that it adds material to the interface when $\lambda > 0$ and takes away material when $\lambda < 0$; it is this term that generates the excess velocity in (26). This is in contrast to the linear term which redistributes the material keeping the total mass unchanged.

An important property of these stochastic growth equations is that they exhibit time and space scaling, that is, the width (standard deviation) of the surface as it evolves over time, exhibits a power law relationship with time and once it reaches a statistical equilibrium, it exhibits a power law dependence on the system size, that is, the spatial averaging length, L, i.e.,

$$w(L) \sim L^H \tag{28}$$

where

$$w(L) = \sqrt{\frac{1}{L}\sum_{i=1}^{L}(h_i - \langle h \rangle)^2}$$

The scaling exponents that arise from these continuum equations fall in universality classes and speak for the fact that not all the “details” but only a few essential factors of the system determine its form. Given the much reported scale invariance in many natural processes, the above model have found extensive application as the governing equations of many physical phenomena (e.g., see references in Barabasi and Stanley, 1995) including landscape evolution (e.g., Somfai and Sander, 1997; Pelletier, 2007; Passalacqua et al., 2006; among others) and rainfall modeling (e.g., Sapoznhikov and Foufoula-Georgiou, 2007). In the next section, a particular observation that relates the Roering et al. (1999) model to the KPZ equation is made.

EMERGENT SCALING FROM A MECHANISTIC MODEL?

A question of great interest is as to whether models that are physically-derived show any specific scale invariance properties. If that were the case, it could serve as an indication that only a few factors dominate the process dynamics and could also provide an additional model diagnostic.

Here we make an interesting observation, namely that the model of Roering et al. (1999) (after the expansion proposed before) takes the form of the KPZ equation where however the coefficient of the nonlinear term λ is not constant but depends on the local curvature and the noise term is missing, i.e, compare equations (21) and (22) to (26). (We term equ (26) a KPZ-like equation). Knowing that the KPZ equation with constant λ and noise gives rise to scaling, the question arises as to whether the Roering et al. model (or the KPZ-like equation) gives also rise to scaling and, if yes, what kind of scaling.

We approach this question both theoretically and by simulation. Theoretically, we note that the randomness in the KPZ-like equation (equ. 21) comes not from an external noise term but from the medium itself (spatial variability of topographic curvatures). As such, we maintain that the KPZ-like model falls under the class of stochastic growth equations with quenched noise (randomness depends on space only and not time; see also Pelletier, 2007 for a recent model that incorporates quenched noise) and specifically the class of pinning-depinning (DPD) models (see Barabasi and Stanley, 1995). Moreover, we observe that the parameter λ can "diverge" at locations of high curvature or at points at which a transition happens from convergent to divergent topography (passing from erosion to deposition). For models with locally divergent λ, Barabasi and Stanley shows that the universality class they belong to shows spatial scaling with exponent $H \simeq 0.63$.

It is interesting to note that the KPZ-like model of Roering et al., with the term λ varying proportional to curvature, can be shown by simulation (starting with a random initial surface) to indeed result in a scaling exponent of approximately 0.6 (unpublished work) which is very close to the theoretical exponent of 0.63 expected from the pinning-depinning class of KPZ models. The importance of this observation is two-fold. First, statistical scaling quantified from observations in a landscape can serve as a model diagnostic of possible classes of models governing the underlying dynamics, since these models must be consistent with the observed scaling. On the other hand, it is important to know whether adopting a specific model carries with it a specific self-similar solution which would imply that the surface self-organizes over time to a statistical steady state for which certain "details" of the system are unimportant. It is noted that the steady-state scaling arising from a purely diffusive model is $H \simeq 0.50$ which can from an additional (statistical) test for hillslope transport laws.

Recent studies have documented the presence of simple-scaling or multi-scaling in several attributes of landscapes including width functions (e.g., see the recent study of Lashermes and Foufoula-Georgiou, 2007) and river corridor width series (see Gangodagamage et al., 2007 who report values of H ranging from 0.3 to 0.8). An important question is what class of evolution equations is rich enough to give rise to dynamic steady-state forms which exhibit scaling consistent with the broad range of scaling observed in nature. This is an important subject of research in several fields of physical science (e.g., see for example, Woyczynski, 1998; Schmidt and Marsan, 2001).

CONCLUDING REMARKS

In this paper, we have proposed the idea that the fractality of landscapes (expressed as power law variability down to very small scales) demands the exploration of non-local constitutive laws which express the flux at a point in terms of elevation gradients in the neighborhood around that point. We suggest that an attractive form of the memory kernel, which dictates how the neigborhood contributes to the local flux, has a power law form. We argue that this form is both consistent with the form of effective scale-dependent coefficients that result from a Large Eddy Simulation (LES) approach and from a mechanistically-derived model of hillslope evolution. Besides, this class of non-local constitutive laws leads to the class of fractional differential models which have been extensively studied in other physical phenomena and make sense physically and mathematically for geomorphology. We argue that the proposed non-local transport laws promise to alleviate the scale-dependence of coefficients which is an important limitation in practical applications.

We suggest that further study of these fractional transport laws is needed in several geomorphic transport processes. Notably, advances can be made in sediment transport modeling (due to the connection of generalized Continuous Time Random Walks, CTRWs, describing particle movement to limiting governing equations, some of which might need fractional derivatives to explain the observed sub- and super-diffusion – see Meltzler and Compte, 2002 -- and also the observed scaling in bed elevation and sediment flux fluctuations e.g., see Nikora and Welsh, 2007; and Sighn et al, 2007), in hillslope transport as discussed in this paper, and in environmental transport on fractal river networks (e.g., Campos et al., 2005; and Bertuzzo et al., 2007).

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